An Introduction to Control Systems

Signals and Systems: 3C1
Control Systems Handout 1
Dr. David Corrigan
Electronic and Electrical Engineering
corrigad@tcd.ie

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- Recall the concept of a System with negative feedback. The output of a dynamic system is subtracted from the input and the resulting signal is passed through the dynamic system.
- This is an example of a closed loop Control System. Control Systems are designed to regulate the output of a system (aka the plant) that otherwise would be sensitive to environmental conditions. The cruise control system in a car is an example of a system where the output speed of a vehicle is to follow a set reference speed.
- The behaviour of the dynamic system is modified through the use of feedback and by cascading the plant with an additional purposely designed system known as a controller.
- Typically the plant is an analogue system (eg. car dynamics) but the controller can either be analogue or digital. If a digital controller is used this implies that extra a/d and d/a stages are added to the control system to allow the controller interface with the dynamic system.

- To cover this topic thoroughly, multiple courses would be required. The goal in 3c1 is to introduce the basic ideas that govern control systems and their design. We will focus on purely analogue controllers. The course builds on the theory outlined in the first 4 handouts of this course.
- In this handout, we will use the example of a cruise control system to demonstrate the benefits of using feedback in control systems.

1 The General Form of a Control System

A control system can be thought of as any system where additional hardware is added to regulate the behaviour of a dynamic system. Control systems can either be *open loop* or *closed loop*. A closed loop system implies the use of feedback in the system. We will see that using feedback allows us more freedom to specify the desired output behaviour of the system.

Perhaps the most common architecture for a closed loop system is shown in Fig 1.

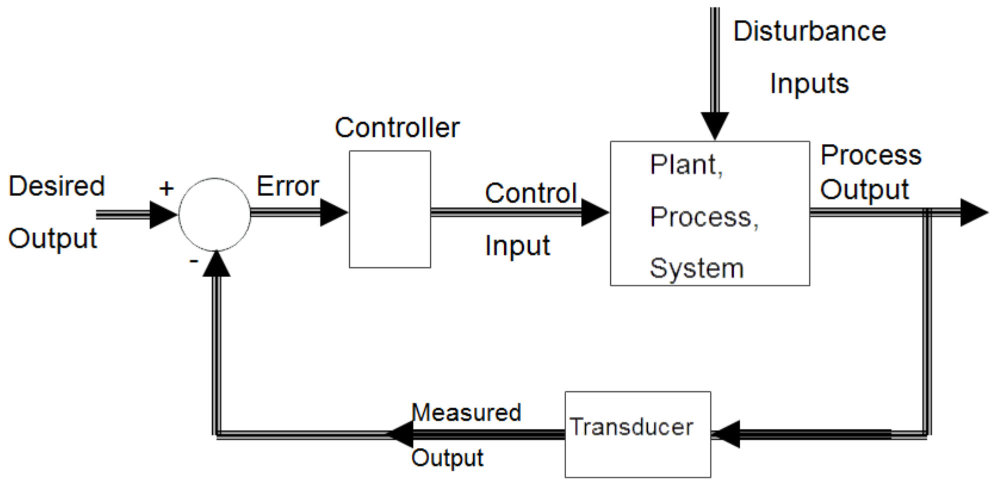


Figure 1: A basic Control System Architecture

We will discuss the various terms in Fig 1 using the example of a car cruise control system

- Desired Output/Reference Input - In systems where it is desired that the output follows the input this signal sets the desired output level. In a cruise control system this would be an electrical signal corresponding to the desired output speed.
- Plant/Process/System - refers to the system we want to control. It generally refers to a physical process which we can either model or measure. The

output of the system is typically not an electrical signal. In the example of the car cruise control system, it encompasses the car dynamics (Newton's Laws). The output signal is the speed of the car.

- Transducer - The role of the transducer is to convert the output signal to an equivalent electrical system (*e.g.* from kinetic energy to electrical energy). This facilitates the comparison of the output signal to the input.
- Error Signal - This an electrical signal that is the difference between the desired and true output signals.
- Controller - This is a purposely designed system to modify the behaviour of the plant.
- Disturbance Input - External/environmental factors that affect plant behaviour (*e.g.* Wind, Gradient).

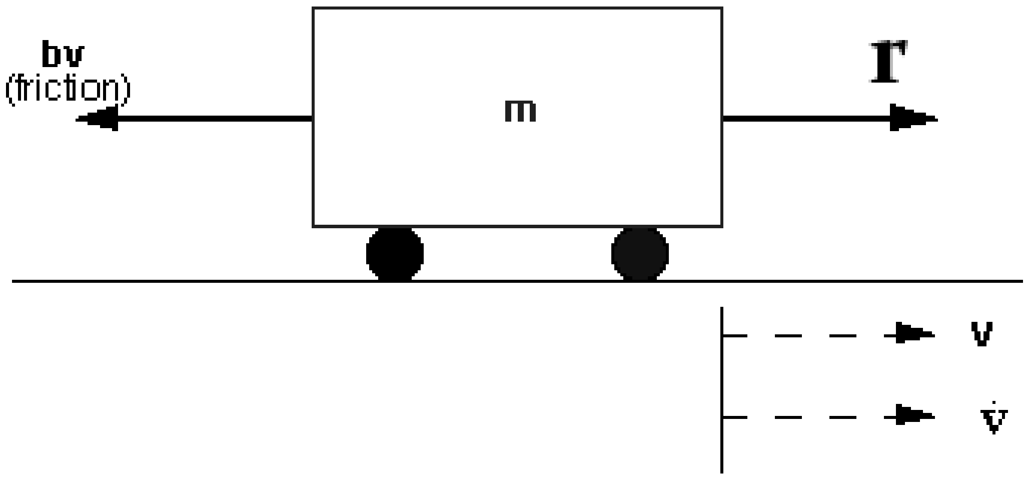
As control engineer we must be able to -

1. model/measure the dynamic behaviour of the plant/system.
2. choose an appropriate controller that allows the system output to meet a list of user designed criteria.

2 Case Study: Car Cruise Control System

We want to design a cruise controller for a car that uses a simple proportional controller. *i.e.* The controller is no more than a simple amplifier ($C(s) = K_p$). The first stage is to model the dynamics of the car. Then we will compare the performance of both open and closed loop controllers.

2.1 Vehicle Dynamics



Here we have an input force signal $r(t)$ (supplied by the engine) and a friction force. The output we are interested is the velocity $v(t)$. The system parameters are the mass m and the friction coefficient b.

To get an ODE for the system we write down Newton's second law

$$\text{Net Force} = \text{mass} \times \text{acceleration}$$

$$r(t) - bv(t) = m\frac{dv}{dt}$$

$$m\frac{dv}{dt} + bv(t) = r(t)$$

2.2 Open Loop Response

The Open Loop Response refers to the response of the system when there is no feedback. The first step is to analyse the step reponse of the plant. We find the system response as before. We will assume 0 initial conditions for the output.

$$\begin{aligned} m\frac{dv}{dt} + bv(t) &= r(t) \\ \Rightarrow \; msV(s) + bV(s) &= R(s) \\ \Rightarrow \; G(s) = \frac{V(s)}{R(s)} &= \frac{1}{ms+b} \end{aligned}$$

So lets say the car has a mass of 1000 Kg, the friction constant is 50 Ns/m and the output force of the engine is 500 N. Then, what is the steady state velocity and what is step response of the system to this engine force?

First of all we can see that

$$G(s) = \frac{1}{1000s + 50} \tag{1}$$

Since we are interested in the step-response to a force of 500 N (*i.e.* $e(t) = 500u(t)$) and assuming 0 initial conditions the output Laplace Transform is

$$V(s) = \frac{500}{s(1000s + 50)}$$

There are many ways to get the steady state velocity. If we use the final value theorem we get

$$\begin{aligned} v_{ss} = \lim_{t\to\infty} v(t) &= \lim_{s\to 0} sV(s) \\ &= \left.\frac{500s}{s(1000s+50)}\right|_{s=0} \\ &= \\ &= \end{aligned}$$

The step response is then

$$
\begin{aligned}
v(t) &= \mathcal{L}^{-1}\left(\frac{500}{s(1000s+50)}\right) = \mathcal{L}^{-1}\left(\frac{10}{s(20s+1)}\right) \\
&= \mathcal{L}^{-1}\left(\frac{10}{s} - \frac{10}{20s+1}\right) \\
&= 10\left(u(t) - e^{-\frac{t}{20}}\right)
\end{aligned}
$$

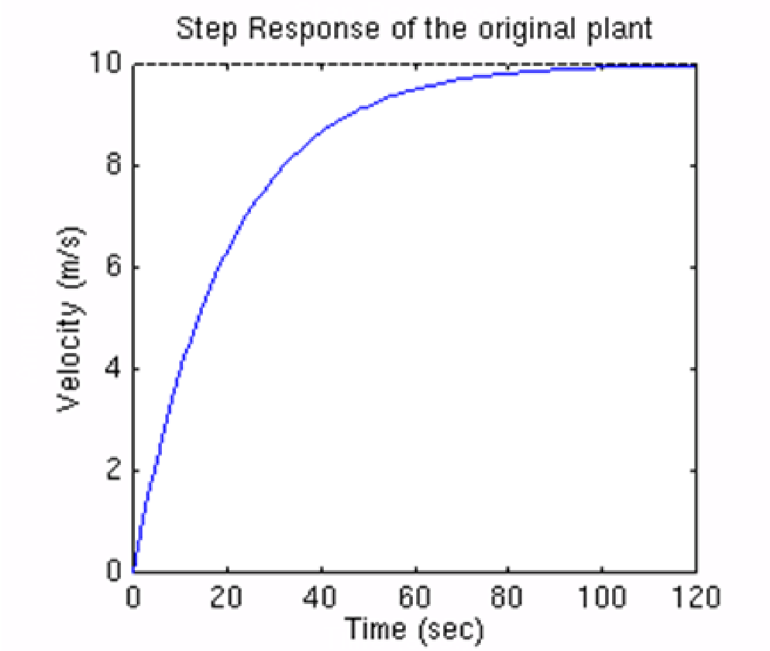


Figure 2: A plot of $v(t)$. This is the open loop response of the system.

We can see from the plot $v(t)$ that it takes a long time for the car to reach the optimum speed. The rise time has various definitions but we define it as the time taken for the output to rise from 10% to 90% for the final output value. In this case, the rise time is approximately 44 seconds. It also takes approximately 100 seconds to reach the optimal speed.

Imagine the desired output speed is 10 m/s. The controller would need to convert the electrical signal corresponding to 10 m/s into a force signal corresponding to 500N. Thus, the gain of the controller is set to 50.

2.3 Closed Loop System

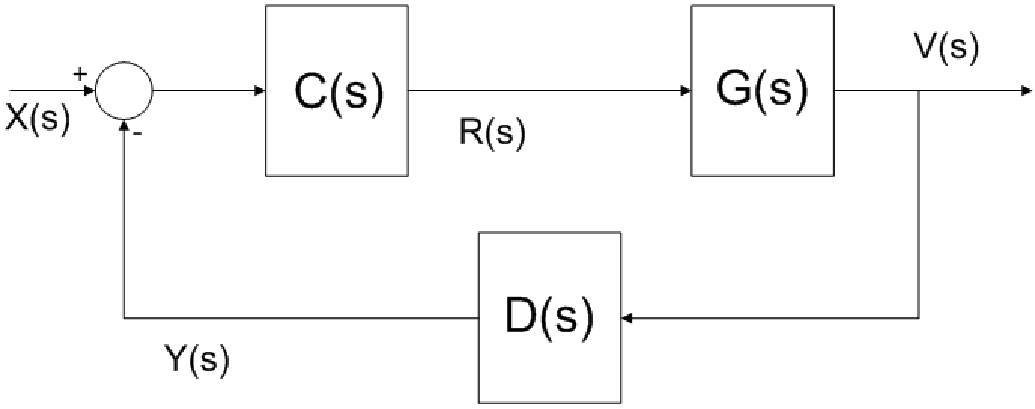


Figure 3: A model for the cruise control system.

We will now show that feedback can be used to reduce the amount of time it takes to reach the desired speed. We use the control system architecture shown in Fig 3 and assume that there are no disturbance inputs. To investigate the design of a feedback control system for the cruise control application we first agree on performance specifications for the control system.

- **Rise Time** $< 5s$ - This gives a 0 to 10 m/s time of approximately 15 seconds.
- **Steady state error** $< 2\%$ - the steady state should be within 2% of the required speed.

The first step is to estimate the transfer function of the control system

$$\begin{aligned} V(s) &= R(s)G(s) = E(s)C(s)G(s) \\ &= C(s)G(s)(X(s) - Y(s)) \\ &= C(s)G(s)X(s) - C(s)G(s)D(s)V(s) \end{aligned}$$

Therefore we can write the transfer function as

$$H(s) = \frac{V(s)}{X(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)D(s)}$$

We are using a simple Proportional Controller $C(s) = K_p$. This implements the control law $R(s) = K_p E(s)$, note also here $D(s) = 1$. This means that the transducer has ideal behaviour $Y(s) = V(s)$.

$$\Rightarrow H(s) = \frac{K_p G(s)}{1 + K_p G(s)} \tag{2}$$

We know from equation 1 that $G(s) = \frac{1}{1000s+50}$. Substituting into equation 2 we get

$$\begin{aligned} H(s) &= \frac{K_p \frac{1}{1000s+50}}{1 + K_p \frac{1}{1000s+50}} = \\ &= \frac{\frac{K_p}{K_p+50}}{\frac{1000}{K_p+50} s + 1} \end{aligned}$$

This is also a first order system. The general form of a 1^{st} order system transfer function can be written as

$$H(s) = \frac{K}{\tau s + 1}$$

where K is the dc gain and τ is the time constant[1].

Therefore

$$\tau = \frac{1000}{K_p + 50} \text{ and } K = \frac{K_p}{K_p + 50} \tag{3}$$

These values can be used to estimate the system specifications

$$\begin{aligned} \text{Rise Time } t_r &\approx 2.2\tau = \frac{2200}{K_p + 50} \\ \text{Steady State Velocity } v_{ss} &= \frac{K_p}{K_p + 50} \times X \end{aligned} \tag{4}$$

where X is the reference speed (*i.e.* 10 m/s), and

$$\text{Steady State Error } = \frac{R - v_{ss}}{R} \times 100 = 100(1 - \frac{K_p}{K_p + 50})$$

Therefore we must choose a value for K_p such that $t_r < 5$ and $\%error < 2\%$.

$$\begin{aligned} t_r &= \frac{2200}{K_p + 50} < 5 \\ &\Rightarrow K_p > 390 \\ \%error &= 100(1 - \frac{K_p}{K_p + 50}) < 2 \\ &\Rightarrow \frac{K_p}{K_p + 50} > 0.98 \\ &\Rightarrow K_p > 2450 \end{aligned}$$

Hence we need to choose $K_p > 2450$ to satisfy all our constraints.

[1]This can be verified easily by examining the form of the impulse response

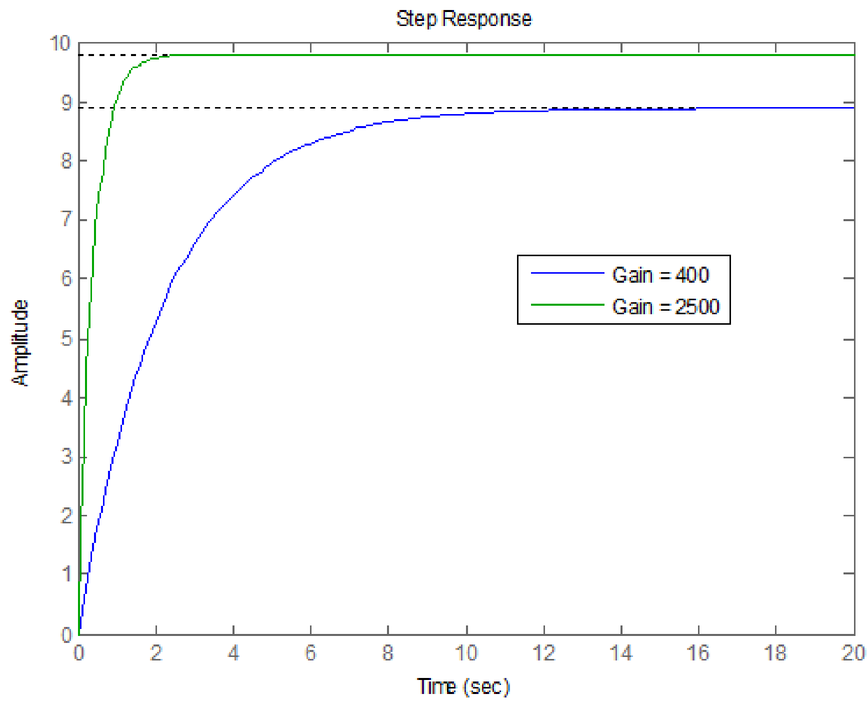


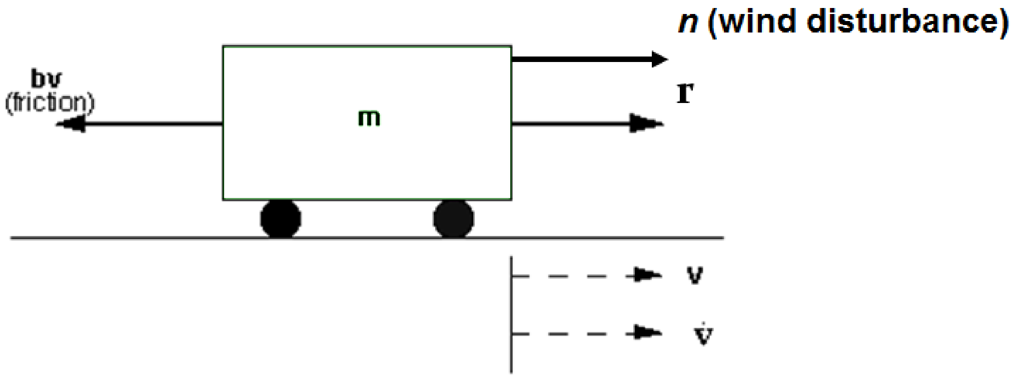
Figure 4: A plot of $v(t)$. This is the closed loop response of the system for gains (K_p) of 400 and 2500.

This design and analysis has shown that a controller gain $K_p = 390$ is sufficient to meet the rise time specification of 5 seconds. But for this gain the steady-state error is over 11% (See Fig 4). Note that the maximum force demanded of the motor is 3900N (confirm).

The error condition can be met if we increase the gain to $K_p = 2500$ but with this level of gain the speed of response has a time constant of 0.4 and a rise time of 0.88sec. However this requires excessive acceleration and the maximum force demanded of the motor is 25000N (confirm) .

As a compromiser well use a gain of $K_p = 750$ which with a rise time of 2.75 seconds (confirm) but with a steady-state error of 6.25%. We will see in the next handout that other types of controller can eliminate steady state error.

2.4 Disturbance Inputs



We will now see how our system responds to a buffet of wind. The wind is modelled as adding an extra force $n(t)$ to the car. Hence we can modify our control system model as follows

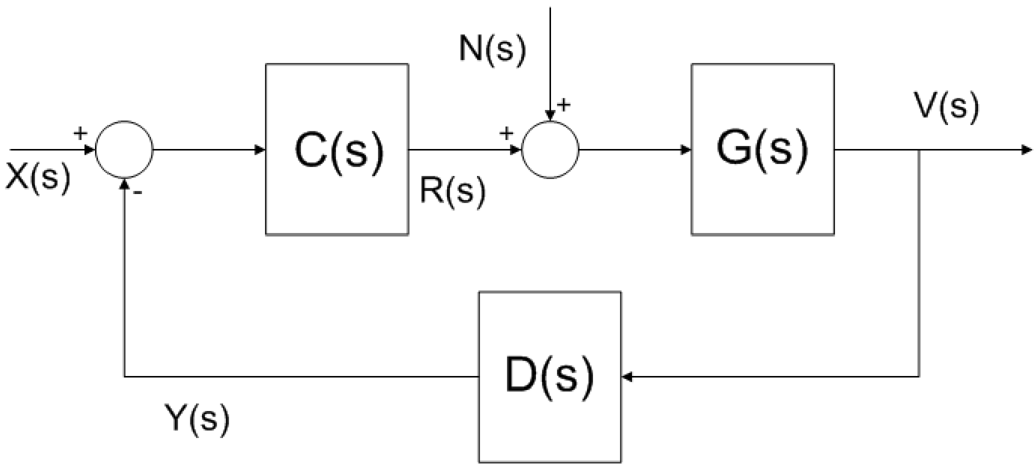


Figure 5: A model for the cruise control system.

To analyse the system response we try to obtain an expression for the output Laplace transform of the form

$$V(s) = H(s)X(s) + F(s)N(s) \tag{5}$$

and so need expressions for $H(s)$ and $F(s)$.

Starting at the plant we get

$$\begin{aligned} V(s) &= G(s)(N(s) + R(s)) \\ &= G(s)C(s)E(s) + G(s)N(s) \\ &= G(s)C(s)(X(s) - D(s)V(s)) + G(s)N(s) \\ \Rightarrow (1 + G(s)C(s)D(s))V(s) &= G(s)C(s)X(s) + G(s)N(s) \end{aligned}$$

$$\Rightarrow V(s) = \frac{G(s)C(s)}{1 + G(s)C(s)D(s)}X(s) + \frac{G(s)}{1 + G(s)C(s)D(s)}N(s)$$

so

$$H(s) = \frac{G(s)C(s)}{1 + G(s)C(s)D(s)}$$

as before and

$$F(s) = \frac{G(s)}{1 + G(s)C(s)D(s)}.$$

Substituting in for $G(s)$, $C(s)$ and $D(s)$

$$H(s) = \frac{K_p}{1000s + K_p + 50}$$

and

$$F(s) = \frac{1}{1000s + K_p + 50}.$$

So for a gain $K_p = 750$ we get the responses to both impulse and step wind disturbances seen in Fig. 6. The step response is given by the inverse Laplace transform of $1/s \times F(s)$ and the impulse response is given by the inverse Laplace transform of $F(s)$.

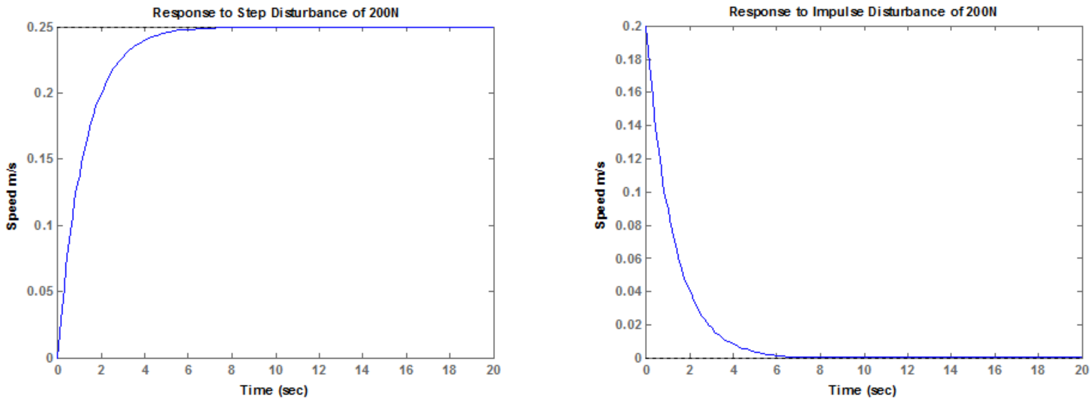


Figure 6: The step (left) and impulse (right) response to a wind force of 200 N. *i.e.* The input signal on the left is $200u(t)$ and on the right is $200\delta(t)$.

From the graphs we can deduce that a constant wind force of 200 N adds an extra 0.25 m/s (0.9 kmph) to the steady state speed and for an impulsive force of 200 N the speed will return to its previous steady state after approximately 7 seconds.

We will now compare this to the response of the open loop system to an equivalent force. The block diagram for the open loop system is seen in Fig 7

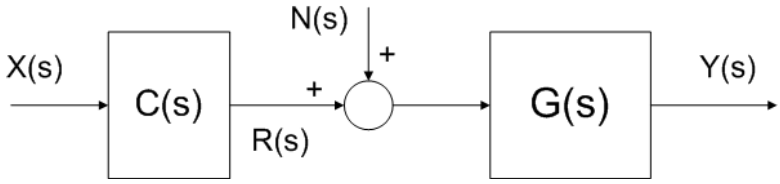


Figure 7: An open loop model for the car.

From the open loop transfer function in Equation (1), the step and impulse responses ($g(t)$ and $h(t)$) to a wind force of 200 N are given by

$$g(t) = \mathcal{L}^{-1}\left(\frac{200}{s} \times \frac{1}{1000s + 50}\right) = 4(1 - e^{-0.05t})$$
$$h(t) = \mathcal{L}^{-1}\left(\frac{200}{1000s + 50}\right) = 0.2e^{-0.05t}$$

and are shown in Fig. 8.

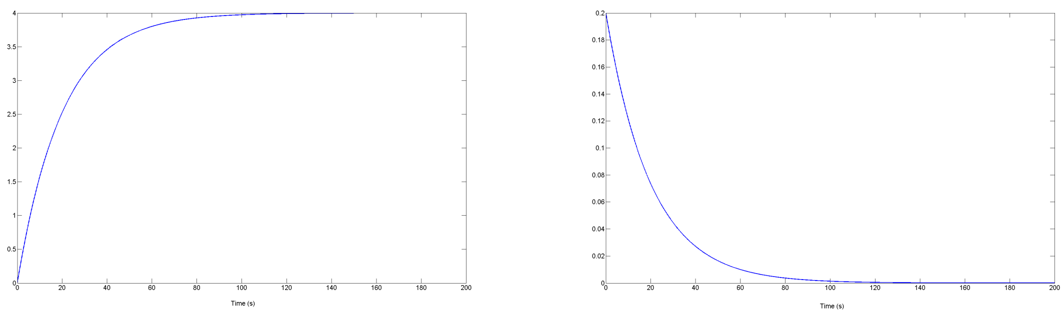


Figure 8: The step (left) and impulse (right) response of the open loop system to a wind force of 200 N. *i.e.* The input signal on the left is $200u(t)$ and on the right is $200\delta(t)$.

Comparing Fig. 8 and Fig. 6 we can see that a constant wind force now adds 4 m/s (14.4 kmph!) to the steady state speed compared with 0.25 m/s for the closed-loop cruise control system. Therefore, the control system is said to reject the disturbance input. For impulsive disturbance the effect on the car velocity lasts much longer for the open loop system (120 s compared to 7s).

Hence the closed-loop control system is more robust to disturbance inputs than the open loop system.

3 Advantages of using Feedback

So far, we have used the example of a cruise control system to highlight some of the advantages of using feedback in dynamic systems. We will do a more general discussion of the advantages of using feedback in control systems.

3.1 Control of the Transient Response

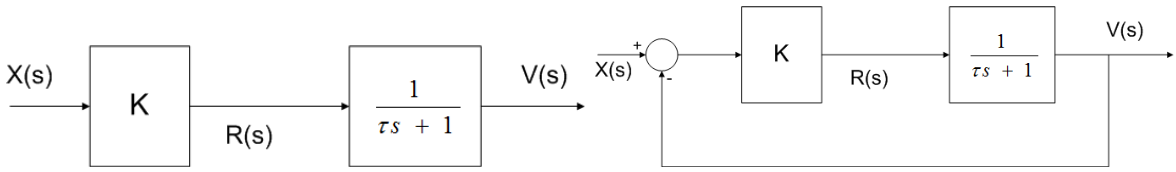


Figure 9: Block Diagrams for open and closed loop 1st Order systems with a proportional controller.

Using feedback allows the transient response of the output to meet design specifications. For example, in the cruise controller it was possible to ensure a rise time of less than 5 seconds by using negative feedback and a proportional controller with a gain of at least 390.

To get a better concept of why this works in general let us look at a first order system in cascade with a proportional controller. This system can operate in an open or closed loop manner as shown in Fig 9. The transfer functions for the open and closed loop systems are given by

$$H_{open}(s) = \frac{K}{\tau s + 1}$$

and

$$H_{closed}(s) = \frac{\frac{K}{\tau s+1}}{1 + \frac{K}{\tau s+1}} = \frac{K}{\tau s + 1 + K} \tag{6}$$

respectively.

The transient responses of these systems to a unit step input can then be obtained in the usual manner as

$$y_{\text{open}}(t) = K(1 - e^{\frac{-t}{\tau}})$$

and

$$y_{\text{closed}}(t) = \frac{K}{1+K}\left(1 - e^{-\frac{1+K}{\tau}t}\right).$$

- In the last section we saw that the rise time is inversely proportional to the decay rate (*i.e.* the factor of t in the exponent). In the open loop system the decay rate is $1/\tau$ which is a constant for a given plant. The controller gain K is typically the only parameter that is allowed to be varied. Adjusting the gain in the open loop system only changes the final value and does not affect the rise time.
- Conversely in the open loop system the decay rate $(1 + K)/\tau$ is a function of the controller gain and so we can adjust the gain to get the desired rise time.

Another way to think about it is in terms of the pole locations of the transfer function.

- In handout 4 (Stability) we showed that the transient response was related to the position of the poles in the transfer function. The closer a pole is to the imaginary axis, the longer it takes the step response of the system to reach its steady state.
- For an open loop system, the positions of the poles is typically constant with respect to the gain of the controller. See Equation (6) for an example. The pole is at $s = -1$ for all K.
- For a closed loop system, the positions of the poles is typically dependent on the value of the gain. In the example shown in (6), the pole is located at $s = -1 - K$.
- In this example, as the value of K increases, the pole gets further away from the imaginary axis and hence the time taken for the step response to reach its steady state decreases.

This remains the case for more complicated plant and controller transfer functions. The use of feedback allows the positions of the poles of the overall transfer function to be modified by changing the value of one or more controller parameters. Hence, we can tune the controller to gain the desired transient response.

3.2 Steady State Error

Feedback (or closed-loop) control also allows the steady state error to be minimised more reliably. This can be seen by using the final value theorem to estimate the steady state error of the open and closed loop systems shown in Fig 10.

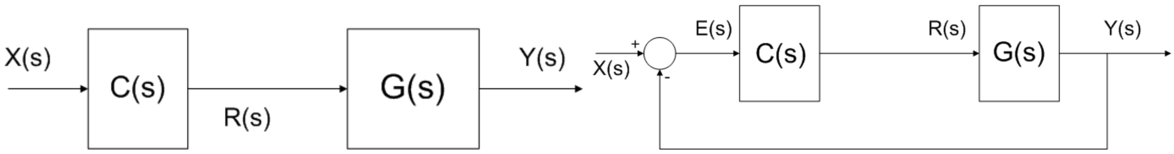


Figure 10: Block Diagrams for open and closed loop 1st Order systems with a proportional controller.

Recall, the transfer functions are

$$H_{open} = C(s)G(s)$$
$$H_{closed} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

The steady state error is the values of the error signal $e(t) = x(t) - y(t)$ as t tends to ∞. Using the final value theorem

$$e_{ss} = \lim_{s \to 0} sE(s) \tag{7}$$

For the open loop system we have

$$E(s) = X(s) - Y(s) = (1 - C(s)G(s))X(s)$$

Then for a unit step $x(t)$,

$$e_{ss} = \lim_{s \to 0} s(1 - C(s)G(s))\frac{1}{s} = 1 - C(0)G(0).$$

For the closed loop system

$$E(s) = X(s) - \frac{C(s)G(s)}{1 + C(s)G(s)}X(s) = \frac{1}{1 + C(s)G(s)}X(s)$$

and so

$$e_{ss} = \lim_{s \to 0} s\frac{1}{1 + C(s)G(s)}\frac{1}{s} = \frac{1}{1 + C(0)G(0)}.$$

At a superficial level it appears that both the open and closed loop system can give a zero steady state error. For the open-loop system a zero error is obtained if we choose the controller $C(s)$ such that $C(0)G(0) = 1$. For the closed loop system $C(0) = \infty$ which means that the controller has a pole at $s = 0$. (*e.g.* $C(s) = 1/s$)

However, the advantage of the closed loop system is that the steady state error is much less sensitive to the choice of controller as well as errors or changes in the plant model $G(s)$. This can be seen by comparing the steady state errors of the open and closed loop systems over a range of values of $C(0)$. Therefore the closed loop system is a more practical way of controlling the steady state error.

3.3 Disturbance Input Rejection

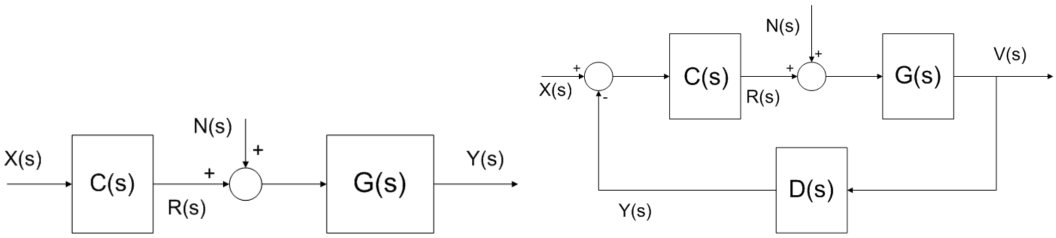


Figure 11: Open and Closed loop Models for disturbance inputs.

Disturbance inputs are modelled as a signal that is added to the system at the input of the plant. We saw in Section 2.4 that the effect of a disturbance input on the system can be modelled as

$$V(s) = H(s)X(s) + F(s)N(s)$$

where $N(s)$ is the disturbance and $F(s)$ is the transfer function with respect to the disturbance.

Examining the expression for $F(s)$ allows us to analyse the reponse of the system to a disturbance. The smaller the magnitude of $F(s)$ the better the capability of the system to reject disturbance inputs. For an open loop system

$$|F_{open}(s)| = |G(s)|$$

and for the closed loop system shown in Fig. 11

$$\begin{aligned} |F_{closed}(s)| &= \frac{|G(s)|}{|1 + G(s)C(s)D(s)|} \\ &\leq \frac{|G(s)|}{1 + |G(s)C(s)D(s)|} \\ &\leq |F_{open}(s)|. \end{aligned}$$

So the magnitude of the disturbance transfer function $F(s)$ is always less for a closed-loop system. Hence the closed-loop system will be better at rejecting disturbance inputs. Furthermore, as the magnitude of the controller transfer function ($|C(s)|$) is increased the sensitivity to disturbance inputs will decrease.