FREQUENCY RESPONSE

Suppose we have a voice recording corrupted with noise (hiss) and a hum from some interfering equipment. The problem is to process this signal to remove the corruption of hum and if possible the noise as well.

We can model the system causing the corruption as follows

\[ y(t) = x(t) + f(t) + e(t) \]

where \( y(t) \) is the observed, degraded signal; \( x(t) \) is the original, clean audio signal, \( f(t) \) is the corrupting hum, and \( e(t) \) is random hiss component. We shall ignore the hiss in this course, analysing that requires an understanding of statistical signal processing (ie signal processing for random signals). So our degradation model is

\[ y(t) = x(t) + f(t) \]

The question is, can we somehow remove the signal \( f(t) \) from \( y(t) \) to leave \( x(t) \)? Think about all you know about LTI systems so far . . .

This is a difficult problem to solve because the two signals \( x(t) \) and \( f(t) \) coincide in time. However, intuitively we can easily perceive these as separate sounds as their pitches are different. Hence, it may be possible to separate these signals if we consider them in terms of their frequency content.

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1This handout is based on the set of notes produced by Prof. Anil Kokaram
Thinking about frequencies

- Most people intuitively appreciate that ‘sounds’ consist of several ‘frequencies’ combined together. In fact, we hear and perceive sounds in terms of **both** frequency and time behaviour.

- Consider some pure sinusoids at $f_1 = 666$ Hz and $f_2 = 166.5$ Hz, $x_1 = \sin(2\pi f_1 t)$, $x_2 = \sin(2\pi f_2 t)$. Intuitively, if we wanted to express these signals in terms of frequency, we would like to be able to show plots as below.

- These are plots of the signal *spectra*, $X_1(f)$, $X_2(f)$. Spectral information can be summarized graphically by the frequency spectrum of $x(t)$ (called $X(\omega)$ or $X(f)$).

- But what about a tone from a guitar? In the lecture you will hear two guitar tones that are in *tune* with the pure sinusoids above. But we know that the guitar sounds different from the pure sinusoid. Logically, you can say that the guitar note must also contain different frequencies.

- In fact we can calculate the frequency spectra of the guitar notes.
You can see some interesting things
- There is more than one frequency present in the signal
- These frequency components seem to be equally spaced in frequency
- The components do not all have the same amplitude

Some questions to answer: what’s the difference between the high guitar note and the low one? What’s the same? There is a lot of background noise (hiss) in the guitar recording, where is that in the spectra?
How does this help?

- Often a signal will contain different frequencies from the hum. As you can see only a small number of frequencies are present in an audio signal. A hum signal might also contain relatively few frequencies. So what if we could design a system to reject the hum frequencies?

- To do this we need to determine how a system responds to different frequencies. The simplest way to do this is to use a sinusoidal input signal \((x(t) = \cos(\omega t))\) and determine the output. What does the output look like? How does the output vary as the frequency \(\omega\) varies?

- FREQUENCY RESPONSE CAN BE MEASURED. Just apply every possible signal input frequency and then measure the output signal.

- By determining how a system responds to a simple sinusoidal input we can determine its response to signals with more complex frequency content.

So

- In this course, we explore the mathematical basis for frequency analysis of signals and systems.

- Frequency analysis of signals is also known as Fourier Analysis and is covered in the next handout. In this handout we will explore the frequency response of systems.

- PLEASE DO NOT IGNORE THE S1 AND S2 LABS, THEY ACTUALLY LEAD YOU THROUGH MANY OF THESE IDEAS AND CAN BE USED AS REVISION TOOLS FOR THE EXAMINATION
ME 62 Omni-Directional Microphone Head

The ME 62 is an omni-directional microphone head suitable for K3 and KSP powering modules. It can be used for reporting, discussions and interviews. The ME 62 is particularly suitable for good reproduction of 'room' ambience and spaced omni stereo recording. Matt black, moulded, scratch-resistant finish.

Features—Benefits

- Omnidirectional pickup pattern
- Minimal inherent self-mix
- Excellent rejection of rattle, wind and handling noise
- Wide frequency response
- High maximum sound pressure level
- Integral pop filter

Technical Data

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<thead>
<tr>
<th>Parameter</th>
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<td>Pickup pattern</td>
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<tr>
<td>Frequency response</td>
<td>20-20,000 Hz ± 2.5 dB</td>
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<tr>
<td>Sensitivity (free field, no load) (1 kHz)</td>
<td>21 mV/Pa ± 2.5 dB</td>
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<tr>
<td>Nominal impedance</td>
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<tr>
<td>Min. terminating impedance</td>
<td>100Ω (with CC)</td>
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<td>Equivalent noise level</td>
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<tr>
<td></td>
<td>CCIR-weighted (CCIR 460-3) 27 dB</td>
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www.sennheiser.co.uk
Specifications:

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<th>Characteristics</th>
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<td>Frequency Response</td>
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<tr>
<td>Sensitivity</td>
<td>87 dB 1V/1m</td>
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<td>Power Handling</td>
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<tr>
<td>Weight</td>
<td>32 pounds</td>
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<tr>
<td>Shielding</td>
<td>Fully Magnetically Shielded</td>
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</table>
1 Frequency Response: Quantitative Analysis

- Want to predict the frequency response of systems analytically. Eventually you will then be able to design a system to have a desired frequency response.

- Basic Result is as follows:

\[
\frac{Y}{X} = |H(j\omega)| \\
\phi = \arg\{H(j\omega)\}
\]

- So we can COMPUTE frequency response \(H(\omega)\) from the transfer function \(H(s)\) by setting \(s = j\omega\) in the transfer function.

- ALSO we can estimate the transfer function by measuring the frequency response.
1.1 Meaning of steady state

TWO USES OF THE TERM STEADY STATE

1. Steady state response to given sinusoidal input.
2. Steady state gain i.e. D.C. Gain (given by $H(0)$).
3. BOTH MEAN WHEN TRANSIENTS HAVE DIED DOWN.
1.2 Expressing Gain in Decibels

It is common to use dB for spectra and frequency response because in many cases it ties in with how we perceive audio and light.

People often get confused about dB because there appears to be 2 different formula for calculating dB. The decibel (dB) is a logarithmic unit that indicates the ratio of power relative to a specified or implied reference power.

\[
R_{dB} = 10 \times \log_{10} \frac{P}{P_{ref}}
\]

If we want to express the gain in dB we have a problem because gain is an amplitude (voltage) ratio and not a power ratio. However, we know that signal power is directly proportional to the square of the amplitude of the signal. Therefore

\[
G_{dB} = 10 \times \log_{10} \frac{Y^2}{X^2} = 20 \times \log_{10} \frac{Y}{X} \quad (1)
\]

So, if dealing with power quantities use the first formula and if dealing with amplitudes use the second formula. We will be dealing primarily with amplitudes.
1.3 Proof of the Basic Result

This applies for rational transfer functions only

\[ H(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s - p_1)^{v_1}(s - p_2)^{v_2} \ldots (s - p_n)^{v_n}} \]

Remember \( \mathcal{L}(e^{-at}) = \frac{1}{s+a} \)

\[ x(t) = e^{j\omega t} \Rightarrow X(s) = \frac{1}{s - j\omega} \]

\[ Y(s) = H(s)X(s) = \frac{n(s)}{d(s)} \times \frac{1}{s - j\omega} = \frac{A_0}{s - j\omega} + \sum_{i=1}^{N} \frac{A_i}{(s - p_i)^{v_i}} \]

By cover up rule:

\[ A_0 = \frac{n(j\omega)}{d(j\omega)} = H(j\omega) \quad \text{(From equation (2) above)} \]

\[ Y(s) = \frac{H(j\omega)}{(s - j\omega)} + \sum_{i=1}^{N} \frac{A_i}{(s - p_i)^{v_i}} \]

\[ y(t) = H(j\omega)e^{j\omega t} + \sum_{i=1}^{N} r_i \frac{t^{(v_i-1)}}{(v_i - 1)!} e^{p_it} \]

Therefore the steady state output:

\[ y_{SS}(t) = H(j\omega)e^{j\omega t} \quad \text{If asymptotically stable.} \]
1.4 WATCH OUT FOR THIS

Given an input \( x(t) = \sin(3t) \) to a system with transfer function \( H(s) \), what is the output of the system \( y(t) \) when it has attained a steady state?

Typical answer:

\[
Y(s) = H(s)X(s) = H(s)\frac{3}{s^2 + 9}
\]

\[
= \ldots \text{ Partial fractions}
\]

So \( y(t) = \ldots \) Use Inverse Laplace Transform.

NOT WRONG BUT LONG!!

Faster way is to recognise that sinusoidal signals passing through LTI systems come out the other end (after transients have died away) scaled by the gain of the system at the frequency of the input signal and lagged by some angle. SO:

\[
y(t) = \left| H(j3) \right| \sin(3t + \arg[H(j3)])
\]

Note that if the question does not say anything about steady state, or it requires you to calculate the transient output as well as the steady state output, then you cannot use this expression, but will have to work out the full response using the full Laplace domain or convolution analysis.
EXAMPLE: Given an input \( x(t) = \sin(3t) \) to a system with transfer function \( H(s) \) (shown below), what is the output of the system \( y(t) \) when it has attained a steady state?

\[
H(s) = \frac{s + 2}{(s + 4)(s + 7)}
\]

we know that the steady state output is going to be a sine wave at the same frequency, but with some phase lag and some different amplitude. The general answer is

\[
y(t) = |H(j\omega)| \sin(\omega t + \arg[H(j\omega)])
\]

where \( \omega \) is the frequency of the input signal.

because \( x(t) = \sin(3t) \Rightarrow \omega = 3 \) Radians per sec (rad sec\(^{-1}\))

so answer is

\[
y(t) = |H(j3)| \sin(3t + \arg[H(j3)])
\]

so now need to work out \( H(j3) \).

\[
H(3j) = \frac{3j + 2}{(s + 4)(s + 7)}
\]

\[
\Rightarrow |H(3j)| = \left| \frac{3j + 2}{(3j + 4)(3j + 7)} \right|
\]

\[
= \frac{|3j + 2|}{|(3j + 4)||3j + 7|} = \frac{\sqrt{3^2 + 2^2}}{\sqrt{3^2 + 4^2}} = 0.95
\]

\[
\arg(H(3j)) = \arg\left( \frac{3j + 2}{(3j + 4)(3j + 7)} \right) = -0.0656 \text{ rad}
\]
2 Bode Diagrams

• Need to summarize the frequency response of a system in a useful way. This is done with plots of Gain and Phase Vs. frequency. (As was shown earlier with examples of response of microphone etc.)

• A Bode diagram of a system consists of 2 graphs GAIN (log) Vs FREQUENCY (Log) and PHASE (lin) vs FREQUENCY (log).

• Why? Because BOTH amplitude and phase of a signal changes as it passes through the system.

• Gain is $|H(j\omega)|$ and Phase is $\arg[H(j\omega)]$.

• Gain is plotted in DECIBELS = $20 \times \log_{10}|H(j\omega)|$ on a linear scale.

• Phase is plotted in Radians or Degrees on a linear scale.

• BOTH PLOTS ARE MADE AGAINST LOG FREQUENCY. EASIEST TO USE LOG PAPER IF POSSIBLE

• Log Frequency is used to allow a large range of frequency to be handled on the same plot.

• By using Log Frequency, drawing Bode diagrams is made simpler than if we had to draw the curves on a linear scale. Also, as our perception of signals obeys log-like frequency laws, it all makes sense to view the system frequency response in a similar way.
• These days, lots of packages like Matlab allow you to plot Bode diagrams automatically. The Matlab function to do this is \texttt{bode(<system transfer function>)}. It’s that easy.

• So why the hell are you going to learn about bode plots?

• Its because you need to interpret these plots, and for design, you need to get a handle on how to change your transfer function to get the desired effect.

\textbf{A really important table}

<table>
<thead>
<tr>
<th>Gain</th>
<th>0.1</th>
<th>$\frac{1}{\sqrt{2}}$</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>$10^{-1.505}$</td>
<td>$10^{0}$</td>
<td>$10^{1}$</td>
<td>$10^{2}$</td>
<td></td>
</tr>
<tr>
<td>$\text{dB} = 20 \log_{10}(\text{Gain})$</td>
<td>-20</td>
<td>-3.01</td>
<td>0</td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>

• A gain of ZERO dBs means a gain of UNITY ! (no change to amplitude of signal as it passes through the system)

• A gain of -ve dBs means attenuation ! (amplitude of signal reduces as it passes through the system)

• A gain of +ve dBs means amplification ! (amplitude of signal increases as it passes through the system)

• A gain of -3dB is significant as it is often thought of as the corner or break frequency of a system.

We will now look at what the Bode plots are for simple transfer functions and show how we can use them to determine the Bode plot for more complex systems.
2.1 WHAT IS BODE PLOT OF $H(s) = (1 + sT)$?

TRICK to fast drawing is to use ASYMPTOTES.

Replace $s$ by $j\omega$:

$$20 \log_{10} \left| 1 + j\omega T \right| = 20 \log_{10} \sqrt{1 + \omega^2 T^2}$$

$$= \begin{cases} 
\approx 0 & \text{dB for } \\
= 20 \log_{10} \sqrt{2} & = 3 \text{dB for } \\
\approx 20 \log_{10}(\omega T) & \text{dB for }
\end{cases}$$

$$\text{arg}[1 + j\omega T] = \tan^{-1}(\omega T)$$

$$= \begin{cases} 
\approx 0 & \text{for } \\
= 45^\circ \ (\text{or } \pi/4 \ \text{rad}) & \text{for } \\
\approx 90^\circ \ (\text{or } \pi/2 \ \text{rad}) & \text{for }
\end{cases}$$

$rac{1}{T}$ is called a ‘breakpoint’, or ‘break frequency’ or ‘corner frequency’ of the asymptote.
PLOTTING BODE DIAGRAM OF $(1 + sT)$

1. Mark point at break frequency $\omega = \frac{1}{T}$, and log(gain) = 3 dB.

2. For $\omega << \frac{1}{T}$ the straight line gain asymptote is a horizontal line of height 0 dB.

3. For $\omega >> \frac{1}{T}$, draw straight line gain asymptote whose gradient is 20 db per Decade. Reason:
   Consider $\omega = \frac{1}{T}$, log(gain of asymptote) = $20 \log_{10}(\omega T) = 20 \log_{10}(1) = 0$ dB.
   Now consider $\omega = \frac{10}{T}$, log(gain of asymptote) = $20 \log_{10}(\omega T) = 20 \log_{10}(10) = 20$ dB.
   So for an increase in frequency of a FACTOR of 10 (a decade !!) the asymptote changes in gain by 20dB. So its slope is 20dB per decade. **Two frequencies $\omega_1$ and $\omega_2$ are separated by one DECADE if $\frac{\omega_1}{\omega_2} = 10$.**

   **THIS IS VERY IMPORTANT TO REMEMBER BECAUSE A DECADE IS THE DISTANCE BETWEEN 1,10; 10,100; 100,1000, AND YOU ARE PLOTTING LOG(GAIN) (Y-AXIS) VS LOG(FREQUENCY) (X-AXIS). ON LOG PAPER THESE DISTANCES (1,10; 10,100 ETC) ARE THE SAME.***
2.1 \((1 + sT)\)

**Bode Diagrams**

Bode plot of \(1+sT\)

- **Frequency (rad/sec)**
  - \(10^{-2}\) to \(10^{2}\)

- **Phase (degrees)**
  - \(0\) to \(100\)

- **db**
  - \(0\) to \(50\)

---

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2.2 WHAT IS BODE PLOT OF $1/(1 + sT)$?

Bode plot of $1/(1 + sT)$

$$
-20 \log_{10} |1 + j\omega T| = -20 \log_{10} \sqrt{1 + \omega^2 T^2} = \begin{cases} 
  \approx 0 & \text{dB for } \omega << (1/T) \\
  = -20 \log_{10} \sqrt{2} = -3 \text{dB for } \omega = (1/T) \\
  \approx -20 \log_{10}(\omega T) & \text{dB for } \omega >> (1/T)
\end{cases}
$$

$$
-\arg[1 + j\omega T] = -\tan^{-1}(\omega T) = \begin{cases} 
  \approx 0 & \text{for } \omega << (1/T) \\
  = -45^\circ \text{ (or } -\pi/4 \text{ rad)} & \text{for } \omega = (1/T) \\
  \approx -90^\circ \text{ (or } -\pi/2 \text{ rad)} & \text{for } \omega >> (1/T)
\end{cases}
$$
BODE PLOT OF $\frac{1}{s}$

Put $s = \frac{1}{j\omega}$

$$\frac{1}{s} = \frac{1}{j\omega}$$

GAIN

$$20 \log_{10} \left| \frac{1}{j\omega} \right| = -20 \log_{10}(\omega) \text{ dB}$$

PHASE

$$\arg \left[ \frac{1}{j\omega} \right] = \tan^{-1} \left( \frac{0}{1} \right) - \tan^{-1} \left( \frac{\omega}{0} \right) = -\tan^{-1}(\infty) = -90^\circ$$

1. The Gain asymptote is just a straight line with gradient $-20$ dB per DECADE. There is no break frequency, its simple enough to draw.

2. Reason :
   Consider $\omega = 0.1$ rad/sec, $\log(\text{gain of asymptote}) = -20 \log_{10}(0.1)$
   $= 20$ dB.
   Consider $\omega = 10$ rad/sec, $\log(\text{gain of asymptote}) = -20 \log_{10}(10)$
   $= -20$ dB.
   So its a straight line of gradient $-20$ dB per decade. Note that it passes through the point $(1, 0)$ Because when $\omega = 1$, gain = 0.
Bode plots of $\frac{1}{s}$
2.3 BODE PLOT OF \( (1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}) \)

Replace \( s \) by \( j\omega \)

\[
20 \log_{10}\left| 1 + 2\zeta \frac{j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2} \right| = 20 \log_{10}\left| (1 - \frac{\omega^2}{\omega_n^2}) + j2\zeta \frac{\omega}{\omega_n} \right|
\]

\[= 20 \log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}
\]

\[
= \begin{cases} 
\approx 0 \text{ dB for } \omega \ll \omega_n \\
= 20 \log_{10}(2\zeta) \text{ dB for } \omega = \omega_n \\
\approx 40 \log_{10}\left(\frac{\omega}{\omega_n}\right) \text{ dB for } \omega \gg \omega_n
\end{cases}
\]

\[
\arg\left[ 1 + 2\zeta \frac{j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2} \right] = \arg\left[ (1 - \frac{\omega^2}{\omega_n^2}) + j2\zeta \frac{\omega}{\omega_n} \right]
\]

\[= \tan^{-1}\left(\frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)
\]

\[= \begin{cases} 
\approx 0^\circ \text{ for } \\
= 90^\circ \text{ for } \\
\approx 180^\circ \text{ for }
\end{cases}
\]
PLOTTING BODE DIAGRAM OF \( \frac{1}{(1+2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})} \)

1. Mark point at break frequency \( \omega = \omega_n \), and \( \log(\text{gain}) = -20 \log_{10}(2\zeta) \) dB. (NEGATIVE BECAUSE THE FUNCTION IS \( 1/(\text{The thing we just looked at on the previous page}) \).)

2. For \( \omega \ll \omega_n \) the straight line gain asymptote is a horizontal line of height 0 dB.

3. For \( \omega >> \omega_n \), draw straight line gain asymptote whose gradient is -40 dB per Decade. Reason:
   Consider \( \omega = \omega_n \),
   \[ \log(\text{gain of asymptote}) = -40 \log_{10}\left(\frac{\omega}{\omega_n}\right) = -40 \log_{10}(1) = 0 \text{dB}. \]
   Now consider \( \omega = 10\omega_n \),
   \[ \log(\text{gain of asymptote}) = -40 \log_{10}\left(\frac{\omega}{\omega_n}\right) = -40 \log_{10}(10) = 40 \text{dB}. \]
   So for an increase in frequency of a FACTOR of 10 (a decade) the asymptote changes in gain by -40dB. So its slope is -40dB per decade. **Two frequencies \( \omega_1 \) and \( \omega_2 \) are separated by one DECADE if \( \frac{\omega_1}{\omega_2} = 10 \).**

4. Note that this high frequency asymptote does indeed pass through the point \((\omega_n, 0)\) on the Bode diagram.
2.3 2nd order system

Bode Diagram

Frequency (rad/sec)
Phase (deg) Magnitude (dB)

Bode Diagram
Frequency (rad/sec)
Magnitude (dB) Phase (deg)

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2.4 Higher Order Systems

Suppose we wish to find the Bode plot for

\[ H(s) = \frac{10(s + 3)}{(s + 100)(1 + s + 25s^2)} \]

The trick is to factorise it into the standard forms we have seen earlier

\[ H(s) = 10(s + 1.5) \times \frac{1}{s + 5} \times \frac{1}{1 + s + 25s^2} \]
\[ = \frac{3}{10} \times \left( \frac{1}{3} s + 1 \right) \times \frac{1}{100 s + 1} \times \frac{1}{1 + s + 25s^2} \]

**AMPLITUDE GAIN :**

\[ 20 \log_{10} |H(j\omega)| = 10 \log_{10} \left( \left| \frac{3}{10} \times \left( \frac{1}{3} j\omega + 1 \right) \times \frac{1}{100 j\omega + 1} \times \frac{1}{1 + j\omega + 25(j\omega)^2} \right| \right) \]
\[ = 20 \log_{10} \left( \left| \frac{3}{10} \right| \right) + \]

Therefore we can get the magnitude plot for \( H(s) \) by adding together the magnitude plots from the individual terms.
PHASE PLOT:

\[
\arg[H(j\omega)] = \arg\left[\frac{3}{10} \times \left(\frac{1}{3}j\omega + 1\right) \times \frac{1}{\frac{1}{100}j\omega + 1} \times \frac{1}{1 + j\omega + 25(j\omega)^2}\right]
\]

\[
= \arg\left(\frac{3}{10}\right) + \arg\left(\frac{1}{1 + j\omega + 25(j\omega)^2}\right)
\]

\[
+ \arg\left(1 + \frac{1}{3}j\omega\right) + \arg\left(\frac{1}{1 + \frac{1}{100}j\omega}\right)
\]

So again we simply add together the phase plots for the different terms.
Higher Order Systems

BODE DIAGRAMS

[Graphs showing Bode plots for magnitude and phase vs. frequency]
3 FILTERS

- Filters are systems which are designed specifically for altering the frequency content of signals.

- A *Low Pass Filter* allows low frequencies to pass through the system unaffected but it stops or ‘attenuates’ high frequencies from passing through the system. The output signal therefore contains either no high frequencies or high frequencies of a smaller amplitude than compared to the input signal. What is ‘low’ or ‘high’ depends on the requirements of the designer.

- A *High Pass Filter* allows high frequencies to pass through the system unaffected but it stops or ‘attenuates’ low frequencies from passing through the system. The output signal therefore contains either no low frequencies or low frequencies of a smaller amplitude than compared to the input signal. What is ‘low’ or ‘high’ depends on the requirements of the designer.

- A *Band Pass Filter* allows a mid-range of selected frequencies to pass through the system unaffected, and attenuates or ‘stops’ all other frequencies outside this range.

- The range of frequencies which is allowed to pass through the filter unaffected is called the ‘pass band’ of the filter. The range which is attenuated is called the ‘stop band’.

- These systems are called ‘filters’ because they affect some frequencies differently from others. This is just like the action of filter paper being used to separate a mixture of particles of different sizes, water from sand say.

- Remember our problem: to make a system to zap some frequencies in a corrupted signal that we think contain the hum? Now we know what we need is to design a **filter** to do the trick.
3.1 Generic Gain of different types of filters

-3dB implies power gain is 1/2

Gain dB

Transition Stopband frequency

3dB bandwidth

Passband

Stopband

Gain dB

frequency

Stopband Transition Band Passband

Gain dB

frequency

Stopband Transition Band Passband

Gain dB

frequency

Stopband Transition Bands Stopband
3.2 An important filter: THE BUTTERWORTH FILTER

- You can spend a lot of time testing functions of $s$ to see what kind of filters you can come up with. In this course we will just look at one popular form.

- The butterworth filter is ‘maximally’ flat in the passband.

- An $N$th order lowpass Butterworth filter has a frequency response with a form as follows:

$$ |G(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2N}} $$

So the second order ($N = 2$) low-pass Butterworth filter has a frequency response of the form

$$ |G(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^4} $$

- Regardless of the value of $N$ the filter gain always has a value of 1 at $\omega = 0$ and passes through $\sqrt{2}/2$ at $\omega = \omega_c$, the 3dB Cutoff frequency.

- It can be made using op-amps in several ways. One example is shown below in which the op-amp is assumed to have infinite gain.

- We will now work through an entire analysis of this filter including deriving the transfer function and then the frequency response. We will then define the bandwidth of the filter.
Figure 1: A Low Pass Filter Network. Suitable choices for the components will result a low pass Butterworth filter.

\( R_1 = R_2 = R_3 = 1\Omega, \ C_1 = 3/\sqrt{2}F, \ C_2 = \sqrt{2}/3F \)

First, find the transfer function \( V_o/V_i \). (Dropping the \((s)\) arguments for brevity).

Sum currents at A

\[
\frac{V_i - V_A}{R_1} = sC_1 V_A + \frac{V_A}{R_2} + \frac{V_A - V_o}{R_3}
\]

Substituting component values gives

\[
\Rightarrow V_i - V_A = \frac{3}{\sqrt{2}} s V_A + V_A + V_A - V_o
\]

(3)

Sum currents at B

\[
\frac{V_A}{R_2} = -\frac{V_o}{1/sC_2} = -\frac{V_o s \sqrt{2}}{3}
\]

(4)

From 3 \( V_i = V_A \left[ s \frac{3}{\sqrt{2}} + 3 \right] - V_o \)

Subst 4 into 3 gives

\[
V_i = \frac{-\sqrt{2} s V_o}{3} \left[ s \frac{3}{\sqrt{2}} + 3 \right] - V_o
\]

\[
= V_o \left[ -s^2 - \sqrt{2} s - 1 \right]
\]

\[
\Rightarrow H(s) = \frac{V_o}{V_i} = \frac{-1}{s^2 + \sqrt{2} s + 1}
\]
Frequency response: put \( s = j\omega \). Is it a Butterworth filter?

\[
\left| H(j\omega) \right| = \left| \frac{-1}{-\omega^2 + \sqrt{2}j\omega + 1} \right|
\]

\[
\left| H(j\omega) \right|^2 = \frac{1}{(1 - \omega^2)^2 + 2\omega^2}
\]

\[
= \frac{1}{1 - 2\omega^2 + \omega^4 + 2\omega^2}
\]

\[
= \frac{1}{1 + \omega^4}
\]

\[
= \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^4}
\]

where \( \omega_c = 1 \). So its a Butterworth filter with \( \omega_c = 1 \) rad/sec.

3dB bandwidth is the range of frequency over which the gain is \( > -3 \)dB.

\[
20 \log_{10} \left| H(j\omega_{3db}) \right| > -3
\]

\[
\Rightarrow 20 \log_{10} \left( \frac{1}{\sqrt{(1 - \omega_{3db}^2)^2 + 2\omega_{3db}^2}} \right) > -3
\]

\[
\Rightarrow \frac{1}{\sqrt{(1 - \omega_{3db}^2)^2 + 2\omega_{3db}^2}} > \frac{1}{\sqrt{-2}}
\]

\[
(1 - \omega_{3db}^2)^2 + 2\omega_{3db}^2 < 2
\]

\[
\Rightarrow 1 + \omega_{3db}^4 < 2
\]

\[
\Rightarrow \omega_{3db}^4 < 1
\]

So 3dB bandwidth of filter is \( \omega_{3db} = 1 \) rad/sec. That means that this filter has a gain GREATER than \( -3dB \) for signals with a frequency from 0 to 1 rad/sec. Above this frequency, the filter attenuates the signal.
4 POLE-ZERO DIAGRAM AND THE FREQUENCY RESPONSE

\[ H(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)} \]

\[ H(j\omega) = \]

\[ |H(j\omega)| \propto \frac{\text{PRODUCT OF DISTANCES FROM } j\omega \text{ TO ZEROS}}{\text{PRODUCT OF DISTANCES FROM } j\omega \text{ TO POLES}} \]

THEREFORE a zero near \( j\omega \) axis ⇒
AND a pole near \( j\omega \) axis ⇒