ESTIMATION OF MULTI-DIMENSIONAL HOMEOMORPHISMS FOR OBJECT RECOGNITION IN NOISY ENVIRONMENTS

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ABSTRACT

We consider the general problem of object recognition based on a set of known templates, where the available observations are noisy. While the set of templates is known, the tremendous set of possible transformations and deformations between the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. We propose a method that reduces the high dimensional problem of evaluating the orbit created by applying the set of all possible transformations in the group to a template, into a problem of analyzing a function in a low dimensional Euclidian space. In this setting, the problem of estimating the parametric model of the deformation is transformed using a set on non-linear operators into a set of equations which is solved by a ***linear*** **least squares solution in the low dimensional Euclidian space. For the case where the signal to noise ratio is high, and the non-linear operators are polynomial compositions, a maximum-likelihood estimator is derived, as well.**

1. INTRODUCTION

This paper is concerned with the general problem of automatic object recognition based on a set of known templates, where the available observations are noisy. While the set of templates is known, the variability associated with the object, such as its location and pose in the observed scene, or its deformation, are unknown *a-priori*, and only the group of actions causing this variability in the observation can be defined. This huge variability in the object signature (for any single object) due to the tremendous set of possible transformations and deformations between the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. In other words, estimation of the transformation of the object with respect to any template in an indexed set is an inherent and essential part of any detection and recognition system. In this paper we address this difficult problem, through a derivation based on a group-theoretic description of the problem, and an algorithmic solution that employs the constraints the observed object and the template must jointly satisfy.

The fundamental settings of the problem are provided in [1]. There are two key elements in a deformable template representation: A typical element (the template); and a family of transformations and deformations which when applied to the typical element produces other elements. The family of deformations considered in this paper is ex termly wide: we consider differentiable homeomorphisms having a continuous and differentiable inverse, where the derivative of the inverse is also continuous.

Thus each template is associated with its orbit, induced by the group action on the template. Hence, given measurements of an observed object (for example, in the form of an image) recognition becomes the procedure of jointly finding the group element and object template that minimize some metric with respect to the observation. Theoretically, in the absence of noise, the solution to the recognition problem is obtained by applying each of the deformations in the group to the template, followed by comparing the result to the observed realization. However, as the number of such possible deformations is infinite, this direct approach is computationally prohibitive. Hence, more sophisticated methods are essential. In addition to the huge variability in the object signature due to the unknown deformations, the observations are also noisy, in general. We analyze the behavior of the proposed solution for estimating the deforming function in the presence of noise. The analysis and the algorithmic solution enable a rigorous treatment of the homeomorphism estimation problem in a wide range of applications.

The center of the proposed solution is a method that reduces the high dimensional problem of evaluating the orbit created by applying the set of all possible homeomorphic transformations in the group to the template, into a problem of analyzing a function in a low dimensional Euclidian space. In general, an explicit modeling of the homeomorphisms group is impossible. We therefore choose to solve this problem by focusing on subsets of the homeomorphisms group which are also subsets of vector spaces. This may be regarded as an approximation the homeomorphism using polynomials, based on the denseness of the polynomials in the space of continuous functions with compact support. In this setting, the problem of estimating the parametric

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model of the deformation is solved by a *linear* system of equations in the low dimensional Euclidian space.

More specifically, consider the problem given by $h(x_1,\ldots,x_n) = g(\boldsymbol{\phi}(x_1,\ldots,x_n))$ where $\boldsymbol{\phi}(x_1,\ldots,x_n) = (\phi_1(x_1,\ldots,x_n),\ldots,\phi_n(x_1,\ldots,x_n))$. In the problem setting considered here h and g are given while $\boldsymbol{\phi}$ should be estimated. In previous papers we analyzed this problem and derived estimation algorithms of the deformation $\boldsymbol{\phi}$, for the case where the transformation is affine [2], and for the case where $\boldsymbol{\phi}$ is a homeomorphism with a differentiable and continuous inverse, [3]. We next briefly summarize the algorithmic solution for the problem of estimating the homeomorphic deformation in the absence of observation noise. To simplify the notation and the accompanying discussion we present the solution for the case where the observed signals are one-dimensional. The derivation for higher dimensions follows along similar lines.

2. PARAMETRIC MODELING AND ESTIMATION HOMEOMORPHISMS WITH A DIFFERENTIABLE AND CONTINUOUS INVERSE

Let R be the one-dimensional Euclidean space, and let the corresponding measure be the standard Lebesgue measure. Consider the general case where $h : X \to Y$ and $g : X \to Y$ are bounded and measurable functions with compact support $X \subset R$, and where $Y \subset R$, such that

$$h(x) = g(\phi(x)) \ . \tag{1}$$

In the following we assume that G is the group of differentiable homeomorphisms such that each element of G has a continuous and differentiable inverse, where the derivative of the inverse is also continuous. Let $C(X)$ denote the set of continuous real-valued functions of X onto itself, where the norm is the standard L_2 norm. By the above assumption every $\phi^{-1}, (\phi^{-1})' \in C(X)$. Since $C(X)$ is a normed separable space, there exists a countable set of basis functions $\{e_i\} \subset C(X)$, such that for every $\phi \in G$,

$$(\phi^{-1})'(x) = \sum_i b_i e_i(x) \ . \tag{2}$$

In other words, it is assumed that every element in the group and its derivative can be represented as a convergent series of basis functions of the separable space $C(X)$. Our goal then, is to obtain the expansion of $\phi^{-1}(x)$ with respect to the basis functions $\{e_i(x)\}$. In practice, the series (2) is replaced by a finite sum, *i.e.*, we have $1 \le i \le m$.

Let $z = \phi(x)$. Then $\phi^{-1}(z) = x$, and hence

$$(\phi^{-1})'(z)dz = dx \tag{3}$$

Let $\{w_p\}_{p=1}^P : Y \to R$ be a set of continuous functions that separate the points of Y. Hence, these functions separate the points of the image of h (and g). As we show next, these functions are employed to translate the identity relation (1) into a set of P equations:

$$\begin{aligned}
\int_{-\infty}^{\infty} w_p(h(x))dx &= \int_{-\infty}^{\infty} w_p(g(\phi(x)))dx \\
&= \int_{-\infty}^{\infty} (\phi^{-1}(z))' w_p(g(z))dz \\
&= \sum_{i=1}^{m} b_i \int_{-\infty}^{\infty} e_i(x) w_p(g(x))dx
\end{aligned}$$

$$p = 1,\ldots,P \tag{4}$$

Rewriting (4) in a matrix form we have

$$\begin{pmatrix} \int w_1 \circ h \\ \vdots \\ \int w_P \circ h \end{pmatrix} = \begin{pmatrix} \int e_1 \ w_1 \circ g & \ldots & \int e_m \ w_1 \circ g \\ \vdots & \ddots & \vdots \\ \int e_1 \ w_P \circ g & \ldots & \int e_m \ w_P \circ g \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \tag{5}$$

We thus have the following theorem:

Theorem: The homeomorphism $\boldsymbol{\phi}$ satisfying the parametric model defined in (1) is uniquely determined iff the matrix

$$\begin{pmatrix} \int e_1 \ w_1 \circ g & \ldots & \int e_m \ w_1 \circ g \\ \vdots & \ddots & \vdots \\ \int e_1 \ w_P \circ g & \ldots & \int e_m \ w_P \circ g \end{pmatrix} \tag{6}$$

is full rank.

Thus, provided that $\{w_p\}_{p=1}^P$ are chosen such that (6) is full rank, the system (5) (in the absence of noise we take $P = m$) can be solved for the parameter vector $[b_1,\ldots,b_m]$. It is clear that in the absence of noise, any set of functions $\{w_p\}_{p=1}^m$ such that (6) is full rank is equally optimal. As we show next the situation is remarkably different in the presence of observation noise.

3. OBSERVATIONS SUBJECT TO ADDITIVE NOISE: THE LEAST SQUARES SOLUTION

In the presence of noise the observed data is given by

$$h(x) = g(\phi(x)) + \eta(x) \ . \tag{7}$$

Assuming that the noise has a zero mean, and that its higher order statistics are known, we first address questions related to issue of the *optimal choice of the set* $\{w_p\}$ *for each template function* g. We begin by adapting the solution derived in the previous section for the deterministic case, to a least squares solution for the model parameters. In the presence of noise the basic equation (4) becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} w_p(h(x))dx &= \int_{-\infty}^{\infty} w_p[g(\phi(x)) + \eta(x)]dx \\
&= \int_{-\infty}^{\infty} w_p[g(z) + \eta(\phi^{-1}(z))](\phi^{-1})'(z)dz \\
&= \int_{-\infty}^{\infty} w_p(g(z))(\phi^{-1})'(z)dz + \epsilon_p^g
\end{aligned} \tag{8}$$

where we define the random variable

$$\epsilon_p^g = \int_{-\infty}^{\infty} \Big\{ w_p[g(z) + \eta(\phi^{-1}(z))] - w_p(g(z)) \Big\} (\phi^{-1})'(z) dz$$
$$= \int_{-\infty}^{\infty} \Big\{ w_p[g(\phi(x)) + \eta(x)] - w_p(g(x)) \Big\} dx \quad (9)$$

Substituting (2) into (8), we obtain the linear system of equations

$$\int_{-\infty}^{\infty} w_p(h(x))dx = \sum_i b_i \int_{-\infty}^{\infty} e_i(x) w_p[g(\phi(x))]dx + \epsilon_p^g$$
$$p = 1, \ldots, P \quad (10)$$

The system (10) represents a linear regression problem where the noise sequence $\{\epsilon_p^g\}$ is non-stationary since its statistical moments depend on the choice of w_p for each p. The regressors are functions of w_p and the template g, and hence are *deterministic*. Provided that the sequence of composition functions $\{w_p\}_{p=1}^P$ is chosen such that the resulting regressors matrix is full rank, the system (10) is solved by a linear least squares method such that the l_2 norm of the noise vector is minimized.

The dependence of the noise sequence $\{\epsilon_p^g\}$ on the choice of w_p suggests that different choices of the composition sequence $\{w_p\}_{p=1}^P$ may provide different solutions. We shall be first interested in systems such that for each p, the linear constraint imposed by w_p is unbiased (and thus the "effective noise" that corresponds to each w_p is zero mean).

3.1. Construction of Unbiased Linear Constraints

Consider the case where we choose $w_p(x) = \sum_k \alpha_k^p x^k$, and the additive noise is white, Gaussian with zero mean and variance σ^2. We next evaluate the mean term, $E\epsilon_p^g$, of the "effective noise", so that a correction term can be introduced such that the non-zero-mean error term ϵ_p^g in (9) is replaced by a zero mean error term. To simplify the notation we will take advantage of linear structure of $w_p(x)$, and analyze only the case where $w_p(x) = x^p$ and the generalization is straightforward. Thus, in this case

$$\epsilon_p^g = \int_{-\infty}^{\infty} \{[g(z) + \eta(\phi^{-1}(z))]^p - g^p(z)\} (\phi^{-1})'(z) dz$$
$$= \int_{-\infty}^{\infty} \sum_{j=1}^{p} \binom{p}{j} g^{p-j}(z) \eta^j(\phi^{-1}(z)) (\phi^{-1})'(z) dz \quad (11)$$

Since the noise is Gaussian with zero mean, all terms involving odd order moments of the noise vanish and hence

$$E\epsilon_p^g = \sum_{j=1}^{[\frac{p}{2}]} \binom{p}{2j} E[\eta^{2j}(\phi^{-1}(z))] \int_{-\infty}^{\infty} g^{p-2j}(z) (\phi^{-1})'(z) dz$$
$$= \sum_i b_i \sum_{j=1}^{[\frac{p}{2}]} \binom{p}{2j} E[\eta^{2j}(\phi^{-1}(z))] \int_{-\infty}^{\infty} g^{p-2j}(z) e_i(z) dz \quad (12)$$

and $E[\eta^{2j}(\phi^{-1}(z))]$ is a constant which is a function of only the index $2j$ of the even order moment, and of σ^2.

Hence, for the case where $w_p(x) = x^p$, the zero-mean-noise version of (10) becomes

$$\int_{-\infty}^{\infty} h^p(x)dx = \sum_i b_i \Big\{ \int_{-\infty}^{\infty} e_i(z) g^p(z) dz$$
$$+ \sum_{j=1}^{[\frac{p}{2}]} \binom{p}{2j} E[\eta^{2j}(\phi^{-1}(z))] \int_{-\infty}^{\infty} g^{p-2j}(z) e_i(z) dz \Big\} + \tilde{\epsilon}_p^g$$
$$(13)$$

where $\tilde{\epsilon}_p^g$ is a zero mean random variable.

Thus the system (13) represents a different linear regression problem where the observation noise is *non-stationary*, but with a zero mean. The regressors are functions of w_p, the template g, and the known statistics of the noise. Hence the regressors are *deterministic*. Provided that the resulting regressors matrix is full rank, the system (13) is solved by a linear least squares method such that $\sum_{p=1}^P |\tilde{\epsilon}_p^g|^2$ is minimized.

4. ANALYSIS OF THE HIGH SNR CASE

In this section we analyze the proposed method assuming $w_p(x) = x^p$, when it is assumed that the signal to noise ratio is high.

To achieve improved numerical stability in the presence of noise, in the following, we consider the normalized version of the system (10), *i.e.*,

$$\frac{1}{\int_{-\infty}^{\infty} g^p(x)dx} \int_{-\infty}^{\infty} h^p(x)dx$$
$$= \sum_i b_i \frac{1}{\int_{-\infty}^{\infty} g^p(x)dx} \int_{-\infty}^{\infty} e_i(x)[g^p(\phi(x))]dx + \bar{\epsilon}_p^g$$
$$p = 1, \ldots, P \quad (14)$$

Since $h(x) = g(\phi(x)) + \eta(x)$ we have under the high SNR assumption that the contribution of high noise powers can be neglected, *i.e.*,

$$h^p(x) \approx g^p(\phi(x)) + p\, \eta(x) g^{p-1}(\phi(x)) \quad (15)$$

Hence, the error term in (9) is approximated under the high SNR assumption by

$$\bar{\epsilon}_p^g = \frac{p}{\int_{-\infty}^{\infty} g^p(x)dx} \int_{-\infty}^{\infty} \eta(x) g^{p-1}(\phi(x)) dx$$
$$= c_p \int_{-\infty}^{\infty} \eta(x) g^{p-1}(\phi(x)) dx \quad p = 1, \ldots, P \quad (16)$$

where we define $c_p = p/\int_{-\infty}^{\infty} g^p(x)dx$. Clearly, $E[\bar{\epsilon}_p^g] = 0,\ p = 1, \ldots, P$. We next evaluate the error covariances of the system, under the high SNR assumption. Let $\tilde{\boldsymbol{\epsilon}}^g = [\tilde{\epsilon}_1^g, \ldots, \tilde{\epsilon}_P^g]^T$, and $\boldsymbol{\Gamma} = E[\tilde{\boldsymbol{\epsilon}}^g(\tilde{\boldsymbol{\epsilon}}^g)^T]$. Thus, the (k,l) element of $\boldsymbol{\Gamma}$ is given by

$$\begin{aligned}
\boldsymbol{\Gamma}_{k,l} &= c_k c_l E[\int_{-\infty}^{\infty} \eta(x) g^{k-1}(\phi(x))dx \int_{-\infty}^{\infty} \eta(y) g^{l-1}(\phi(y))dy] \\
&= c_k c_l \sigma^2 \int_{-\infty}^{\infty} g^{k+l-2}(\phi(x))dx \\
&= c_k c_l \sigma^2 \int_{-\infty}^{\infty} (\phi^{-1}(z))' g(z)^{k+l-2} dz \\
&= c_k c_l \sigma^2 \sum_i b_i \int_{-\infty}^{\infty} e_i(z) g(z)^{k+l-2} dz \qquad (17)
\end{aligned}$$

Rewriting (17) in matrix form we have

$$\boldsymbol{\Gamma} = \sigma^2 \sum_i b_i \mathbf{u}_i \qquad (18)$$

where

$$\mathbf{u}_i = \begin{pmatrix} c_1^2 \int_{-\infty}^{\infty} e_i(z)dz & \ldots & c_1 c_P \int_{-\infty}^{\infty} e_i(z) g(z)^{P-1} dz \\ \vdots & \ddots & \vdots \\ c_1 c_P \int_{-\infty}^{\infty} e_i(z)dz & \ldots & c_P^2 \int_{-\infty}^{\infty} e_i(z) g(z)^{2P-2} dz \end{pmatrix} \qquad (19)$$

5. MAXIMUM LIKELIHOOD ESTIMATION OF THE HOMEOMORPHISM

As we illustrate below, the LS algorithm for estimating the homeomorphism parameters is accurate and computationally simple as it requires only the solution of a set of linear equations. In particular there is no need for an iterative solution. Nevertheless, in cases where the performance of this algorithm is not sufficient, it can serve to initialize the more complex maximum likelihood estimator (MLE) of the parameters, which we derive in this section.

Assuming the observation noise $n(x)$ is Gaussian, we have under the high SNR assumption that $\bar{\boldsymbol{\epsilon}}^g$ is a zero mean Gaussian random vector with covariance matrix $\boldsymbol{\Gamma}$ given in (18). Rewriting (10) in a matrix form, assuming the high SNR assumption we have

$$\mathbf{h} = \mathbf{G}\mathbf{b} + \bar{\boldsymbol{\epsilon}}^g \qquad (20)$$

where $\mathbf{h} = [\int_{-\infty}^{\infty} w_1(h(x))dx, \ldots, \int_{-\infty}^{\infty} w_P(h(x))dx]^T$, and $\mathbf{G}$ is a $P \times m$ matrix such that $\mathbf{G}_{k,l} = \int_{-\infty}^{\infty} e_k(x) w_l[g(x)]dx$.

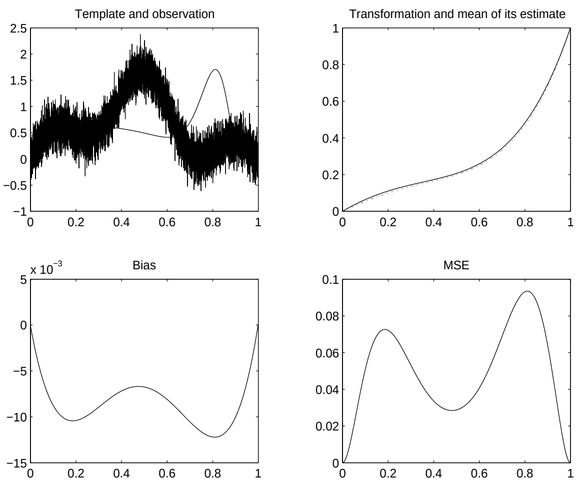

Figure 1: The deformation function and statistics of the least-squares estimate. Clockwise from top-left: The template and a noisy observation; The deforming function (solid line)and the mean of its estimate (dotted line); The bias in estimating the deformation function; The mean squared error in estimating the deformation function.

Hence the log-likelihood function for the observation vector $\mathbf{h}$ is given by

$$\begin{aligned}
\log p(\mathbf{h}; \mathbf{b}) &= \\
-\frac{m}{2}\log(2\pi) - \frac{1}{2}\log(|\boldsymbol{\Gamma}|) &- \frac{1}{2}(\mathbf{h} - \mathbf{G}\mathbf{b})^T \boldsymbol{\Gamma}^{-1}(\mathbf{h} - \mathbf{G}\mathbf{b}) \qquad (21)
\end{aligned}$$

The MLE of the field parameters is found by maximizing $\log p(\mathbf{h}; \mathbf{b})$ with respect to the model parameters $\mathbf{b}$. Since this objective function is highly nonlinear in the problem parameters, the maximization problem cannot be solved analytically and we must resort to numerical methods. In order to avoid the enormous computational burden of an exhaustive search, we use the following two-step procedure. In the first stage we obtain a suboptimal initial estimate for the parameter vector $\mathbf{b}$ by using the algorithm described in Section 3. In the second stage we refine these initial estimates by an iterative numerical maximization of the log likelihood function. In our experiments we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton optimization method [4]. This algorithm requires evaluation of the first derivative of the objective function at each iteration:

$$\begin{aligned}
\frac{\partial \log p(\mathbf{y}; \mathbf{b})}{\partial \mathbf{b}_i} &= -\frac{1}{2}\mathrm{tr}\{\boldsymbol{\Gamma}^{-1}\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\theta}_i}\} + \mathbf{G}_i^T \boldsymbol{\Gamma}^{-1}(\mathbf{h} - \mathbf{G}\mathbf{b}) \\
&+ \frac{1}{2}(\mathbf{h} - \mathbf{G}\mathbf{b})^T \boldsymbol{\Gamma}^{-1} \frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\theta}_i} \boldsymbol{\Gamma}^{-1}(\mathbf{h} - \mathbf{G}\mathbf{b})^T \qquad (22)
\end{aligned}$$

where $\mathbf{b}_i$ is the ith element of $\mathbf{b}$, $\mathbf{G}_i$ denotes the ith column of $\mathbf{G}$, and

$$\frac{\partial \boldsymbol{\Gamma}}{\partial \boldsymbol{\theta}_i} = \sigma^2 \mathbf{u}_i \qquad (23)$$

6. NUMERICAL EXAMPLES

In this section we present numerical examples to illustrate the operation and performance of the proposed model, and parameter estimation algorithm. In the first example we consider the case where the template function is given by $g(x) = \sin(\pi x) + 0.2\sin(2\pi x) - 0.3\sin(3\pi x) + 0.4\sin(5\pi x)$ and the deformation is $\phi(x) = 0.6x - 0.4x^2 - 0.8x^3 + 1.6x^4$. The observed signal is $h(x) = g(\phi(x)) + n(x)$, where the noise is normally distributed with zero mean and variance of 0.04. The number of available samples of both the template and the noisy observation is 10000, so that the effects of both integration errors as well as of low sampling rates (and the resulting need for interpolating the data) are negligible. We illustrate the performance of the proposed solution using Monte Carlo simulations. The experimental results are based on 500 independent realizations of the observed signal. The estimator employed is the least-squares estimator. The top-left plot of Fig. 1 shows the template and a single noisy observation. The top-right plot shows the deforming function and the mean value of its estimates. The two plots on the bottom depict the bias and the mean squared error in estimating the deformation as a function of x (the deformation takes place along the x-axis).

The next example illustrates the operation of the proposed algorithm on an image of a real object. The image dimensions are 1170×1750. The top image in Figure 2 depicts the original image of the aircraft, which is also employed as the template. In order to be able to evaluate the performance of the method the image of the object is then deformed, and a zero mean Gaussian observation noise is added – to obtain the simulated noisy observation of the aircraft. See the middle image. The deforming function (which takes place only along the x-axis) is depicted using a solid line in Figure 3 along with the estimate (dotted line) obtained by applying the proposed solution to the noisy observation shown in Figure 2. Finally, the estimated deformation is applied to the original template in order to obtain an estimate of the deformed object (lower image in Figure 2) which can be compared with the deformed noisy object shown in the middle image.

7. REFERENCES

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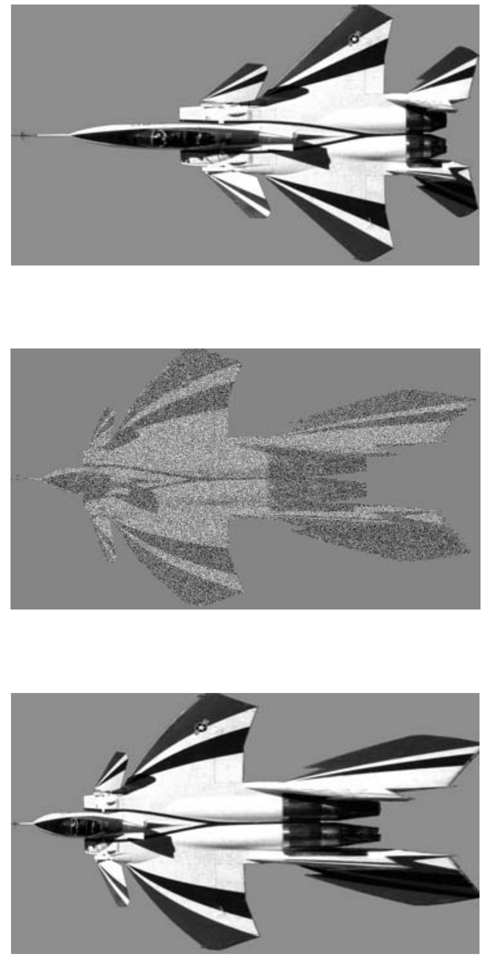

Figure 2: From top to bottom: Template; Noisy observation on the deformed object; Estimated deformed object

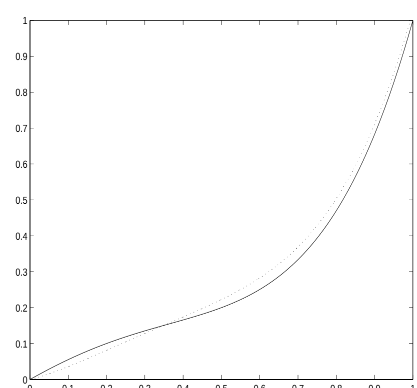

Figure 3: The deformation function and it estimates.