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Spectral representation and asymptotic properties of certain deterministic fields with innovation components

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Abstract. In this paper we derive the spectral and ergodic properties of a special class of homogeneous random fields, which includes an important family of evanescent random fields. Based on a derivation of the resolution of the identity for the operators generating the homogeneous field, and using the properties of measurable transformations, the spectral representation of both the field and its covariance sequence are derived. A necessary and sufficient condition for the existence of such representation is introduced. Using an analysis approach that employs the solution to the linear Diophantine equations, further characterization and modeling of the spectral properties of evanescent fields are provided by considering their spectral pseudo-density function, defined in this paper. The geometric properties of the spectral pseudo-density of the evanescent field are investigated. Finally, necessary and sufficient conditions for mean and strong ergodicity of the first and second order moments of these fields are derived. The analysis, initially carried out for complex valued random fields, is later extended to include the case of real valued fields.

1. Introduction

The problem of linear prediction of homogenous random fields in two or more variables (and in general, fields defined on a compact Abelian group whose dual has a complete linear order compatible with the group structure) was first introduced rigorously in Helson and Lowdenslager [15]. The problem of defining past and future on the two-dimensional lattice (i.e., $\mathbb{Z}^2$) was defined in [15] in terms of "half plane" total-ordering. One of the main results provided in [15] is a generalization of Szegö's formula for the prediction error variance. Further analysis of the prediction problem led to a generalization of the Wold decomposition [16]. Cheng and Houdré [2] generalized some of the results to random fields of random variables with finite p-th moment ($1 < p < \infty$), with an appropriate extended definition of homogeneity; the notion of orthogonality was replaced by the Birkhoff-James

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orthogonality. In Suciu [23] the Wold decomposition is studied for a semigroup of isometries which is not completely ordered.

The well known Wold decomposition of stationary complex valued processes indexed by $\mathbb{Z}$ (see Doob [5, p. 576]) contains two *stationary* parts: the purely-indeterministic process (which is producing the innovations) and the deterministic process. This decomposition can be equivalently reformulated using spectral notations: the spectral measure of the purely-indeterministic process is absolutely continuous with respect to the Lebesgue measure, and the spectral measure of the deterministic process is singular (i.e., the spectral measures of these orthogonal components yield the Lebesgue decomposition of the spectral measure of the process). When we consider homogenous random fields indexed by other groups (like those indexed by $\mathbb{Z}^2$) we obtain a Wold decomposition with respect to any given total order on the group. When the group is not $\mathbb{Z}$ (like $\mathbb{R}$ or $\mathbb{Z}^2$) the deterministic process can have as a direct summand a deterministic process of a special type, the evanescent process. Evanescent processes were first introduced in [16] (on $\mathbb{R}$). In Korezlioglu and Loubaton [18], "horizontal" and "vertical" total-orders and the corresponding horizontally and vertically evanescent components of a homogeneous random field on $\mathbb{Z}^2$ are defined. In Kallianpur [17], as well as in Chiang [3], similar techniques are employed to obtain four-fold orthogonal decompositions of regular (non-deterministic) homogeneous random fields. In Francos et al. [8] this decomposition of random fields on $\mathbb{Z}^2$ was further extended. This is done by considering *all* the rational nonsymmetrical half plane linear orders (RNSHP), each inducing a different partitioning of the two-dimensional lattice into two sets by a broken straight line of rational slope. Clearly, there are countably many such linear orders. Cuny [4] proved recently that the Wold decomposition of a regular random field into purely-indeterministic and deterministic components is the same for all RNSHP orders. The decomposition in [8] asserts that we can represent the deterministic component of the field as a mutually orthogonal sum of a "half-plane deterministic" field and a countable number of evanescent fields. The half-plane deterministic field has no innovations, nor column-to-column innovations, with respect to any RNSHP linear order. Each evanescent field spans a Hilbert space identical to the one spanned by its column-to-column innovations, where the column-to-column innovation at each lattice point is defined as the difference between the actual value of the deterministic field and its projection on the Hilbert space spanned by the deterministic field samples in all previous columns. (Clearly, the term "column" is redefined for each definition of the linear order). Each of the evanescent fields can be revealed only by using the corresponding linear order. This decomposition yields a corresponding spectral decomposition, i.e., we can decompose the spectral measure of the deterministic part into a countable sum of mutually singular spectral measures. In [4] this decomposition has been extended to the case where the measurements are random vectors, rather than random scalars (with the results in [8] as a special case).

Evanescent random fields are of great theoretical and practical importance. Such fields arise quite naturally in problems of texture modeling, estimation, and coding of images (see, *e.g.*, [9] and the references therein), and in space-time adaptive processing of airborne radar data, (see [7] and the references therein). Image indexing and retrieval methods that are based on the Wold decomposition of homogeneous random fields have become the state-of-the-art texture-based image retrieval methods (see, *e.g.*, [20], [21] and the references therein).

In this paper we always assume that we have a known probability space $(\Omega, \mathbb{P})$, the space of our discussion. Let $\{x(n)\}_{n=-\infty}^{\infty} \subset L_2(\Omega, \mathbb{P})$ be a weakly stationary process, i.e. (see [5]), there exists a unitary operator U, defined on $\mathcal{H}$, the closed linear manifold (c.l.m.) spanned by $\{x(n) : n \in \mathbb{Z}\}$, such that $x(n) = U^n x(0)$. From the spectral representation theorem we have a stochastic set function w induced by a projection valued measure E defined on the Borel sets of the unit circle Γ; thus, $w(\cdot) = E(\cdot)x(0)$ and

$$x(n) = U^n x(0) = \int_\Gamma z^n dw(z).$$

We now recall some definitions. A homogeneous random field $\{z(n, m)\}$ is called *deterministic* with respect to the lexicographic order if for every (n, m) we have $z(n, m) \in c.l.m.\Big[\{z(k, l) : k < n, l \in \mathbb{Z}\} \cup \{z(n, l) : l < m\}\Big]$. We say that the field $\{z(n, m)\}$ has vertical *column-to-column innovations* if $I(n, m) := z(n, m) - \hat{z}(n, m)$ (the *innovation*) is not 0, where $\hat{z}(n, m)$ is the orthogonal projection of $z(n, m)$ on the closed subspace generated by $\{z(k, l) : k < n, l \in \mathbb{Z}\}$. When $z(n, m)$ is deterministic, the vertical evanescent component $z_e(n, m)$ is the orthogonal projection of $z(n, m)$ on the closed subspace generated by the (orthogonal!) column-to-column innovations $\{I(k, l) : k \leq n,\ l \in \mathbb{Z}\}$. Column-to-column innovations and evanescent components with respect to any RNSHP order are defined similarly (see §2).

The random field $e(n, m) = x(n)\lambda^m$, where $\{x(n) : n \in \mathbb{Z}\}$ is a weakly stationary (complex valued) purely-indeterministic process and λ is a fixed known point on the unit circle, was essentially considered in [8] as an important typical example of an evanescent homogeneous random field with column-to-column innovations relative to the usual lexicographic order. In this paper, we consider fields given by $e(n, m) = x(na + mb)\lambda^{nc+md}$ where $\{x(n)\}$ is only assumed to be weakly stationary, and a, b, c, and d are integers satisfying $|ad - bc| = 1$ and $ab \neq 0$. We show that when $\{x(n)\}$ is purely-indeterministic, $\{e(n, m)\}$ is an evanescent random field with respect to the order induced by a and b.

2. The evanescent component

A homogenous random field $z(n, m)$ can be obtained by multiplying two independent one-dimensional weakly stationary sequences $\{x(n)\}$ and $\{y(m)\}$. As a special case, we look at the field $e(n, m) = x(n)\lambda^m$, with $\{x(n)\}$ purely-indeterministic, which was considered in [8] (see the introduction). This field is obviously deterministic with respect to the lexicographic order. If $\hat{x}(n)$ denotes the orthogonal projection of $x(n)$ on the closed subspace generated by $\{x(k) : k < n\}$, then $\hat{e}(n, m) = \hat{x}(n)\lambda^m$. Indeed, by definition

$$\hat{x}(n)\lambda^m \in c.l.m.\{x(k) : k < n\} = c.l.m.\{e(k, l) : k < n, l \in \mathbb{Z}\}.$$

This and the Wold decomposition of $\{x(n)\}$ yield

$$e(n,m) - \hat{x}(n)\lambda^m = [x(n) - \hat{x}(n)]\lambda^m \perp c.l.m.\{e(k,l) : k < n, l \in \mathbb{Z}\}.$$

The uniqueness of the orthogonal projection yields that $\hat{e}(n,m) = \hat{x}(n)\lambda^m$, and $\|e(n,m) - \hat{e}(n,m)\| = \|x(n) - \hat{x}(n)\| > 0$ if and only if $\{x(n)\}$ is regular (not deterministic), which is the case since we assumed that $\{x(n)\}$ is purely-indeterministic. Thus, the deterministic field $\{e(n,m)\}$ has vertical column-to-column innovations.

Rotating the previous field by 90°, i.e., by taking $e(n,m) = x(m)\lambda^n$, we obtain a field with column-to-column innovations relative to the orthogonal lexicographic order (horizontal column-to-column innovations). A "rotation" of the usual lexicographic order, such that the RNSHP is delimited by a line with rational slope [8], leads to a generalization of the above field model, which is

$$e(n,m) = x(na + mb)\lambda^{nc+md} \tag{1}$$

We assume throughout that $\{x(n)\}$ is a weakly stationary complex valued process, and a, b, c, and d are integers with $ab \neq 0$ satisfying $|ad - bc| = 1$.

Proposition 2.1. *A field of the form (1) is deterministic with respect to any RNSHP. It is evanescent, with respect to the order induced by a and b, if and only if* $\{x(n)\}$ *is purely-indeterministic.*

Proof. To obtain the order of RNSHP induced by any non-zero pair of integers (α, β), we define the past $P_{\alpha,\beta}$ by

$$P_{\alpha,\beta} = \{(n,m) \in \mathbb{Z}^2 : n\alpha + m\beta < 0\text{ , or } n\alpha + m\beta = 0 \text{ and } m \leq 0\}. \tag{2}$$

Then $P = P_{\alpha,\beta}$ satisfies

$$(i)\ P \cap (-P) = \{0\}, \quad (ii)\ P \cup (-P) = \mathbb{Z}^2, \quad (iii)\ P + P \subset P\ (usual\ addition).$$

By (i)–(iii), P induces on $\mathbb{Z}^2$ a linear order, which is defined by $(p,q) \preceq (n,m)$ if and only if $(p-n, q-m) \in P$. Note also that P contains a maximal element, namely $(0,0)$.

Now we consider the order induced by $P = P_{a,b}$, where a and b are the parameters of e in (1). Put $t = \text{sign}(a)$; then $(n + tb, m - ta) \prec (n,m)$. Since $e(n,m) = e(n + tb, m - ta)\lambda^t$, the field $\{e(n,m)\}$ is deterministic with respect to the RNSHP determined by P (physically, for $k = na + mb$, we have a fixed choice of the random part $x(k)$ along the line through (n,m) with slope $-a/b$). By geometrical considerations, for any non-zero pair of integers α and β we have $P_{\alpha,\beta} \cap P_{a,b} \neq \emptyset$. In fact, the intersection contains a ray (half line) of the boundary line of $P_{a,b}$. Therefore e has no innovations with respect to any RNSHP. Define the closed linear manifolds

$$\mathcal{E}_{n,m} = c.l.m.\{e(p,q) : (p,q) \preceq (n,m)\}, \tag{3}$$

where $\preceq$ is the order induced by $P_{a,b}$, and

$$\mathcal{X}_n = c.l.m.\{x(m) : m \leq n\}.$$

Since $(p,q) \preceq (n,m)$ implies $pa + qb \leq na + mb$ we have $\mathcal{E}_{n,m} \subset \mathcal{X}_{na+mb}$.

Without loss of generality (see the discussion later) we assume that $ad - bc = 1$. Since for an integer $k \geq 0$ we have $(n - kd, m + kc) \preceq (n, m)$, we conclude that $\mathcal{X}_{na+mb} \subset \mathcal{E}_{n,m}$, and hence $\mathcal{X}_{na+mb} = \mathcal{E}_{n,m}$. Therefore $\cap_{n,m}\mathcal{E}_{n,m} = \cap_{n,m}\mathcal{X}_{na+mb} = \cap_s \mathcal{X}_s$, which means that $\{x(n)\}$ and $\{e(n, m)\}$ have the same remote pasts. By the representation of $\{e(n, m)\}$, it is orthogonal to its remote past if and only if $\{x(n)\}$ is orthogonal to its remote past. The latter is possible if and only if $\cap_s \mathcal{X}_s = \{0\}$, and that is true if and only if $\{x(n)\}$ is purely-indeterministic (see [5, p. 579], [11, p. 75]). Hence the deterministic field $\{e(n, m)\}$ is orthogonal to its remote past (i.e., is evanescent [16, p. 181]) if and only if $\{x(n)\}$ is purely-indeterministic. □

The following was noticed by M. Lin, by applying Proposition 2.1.

Corollary 2.2. *The following are equivalent:*

(i) for every a, b, c, and d with $|ad - bc| = 1$, the field $x(na + mb)\lambda^{nc+md}$ is evanescent with respect to the order induced by a and b.
(ii) the field $x(n)\lambda^m$ is evanescent with respect to the usual lexicographic order.
(iii) $\{x(n)\}$ is purely-indeterministic.

Remark. The above model of an evanescent field is not the most general one, [16], [18], [8]; for an example see [4].

Column-to-column innovations are defined for any RNSHP as follows:

Definition 2.1. *Let $\hat{z}(n, m)$ be the orthogonal projection of a homogenous random field $z(n, m)$ on the closed subspace generated by*

$$\{z(p, q) : (p - n)\alpha + (q - m)\beta < 0\}.$$

We say that $\{z(n, m)\}$ has column-to-column innovations with respect to the order induced by α and β if $z(n, m) - \hat{z}(n, m)$ (the innovation) is not 0.

The evanescent component of $\{z(n, m)\}$ with respect to an RNSHP order is determined by its the column-to-column innovations with respect to that order [8] (for vertical column-to-column or horizontal row-to-row innovations see [18]).

It is easy to check (using the same technique as for the usual lexicographic order) that $\hat{e}(n, m) = \hat{x}(na + mb)\lambda^{nc+md}$, and that $\{e(n, m)\}$ has column-to-column innovations with respect to the order induced by a and b if and only if $\{x(n)\}$ is not deterministic. Since $P_{\alpha,\beta} \cap P_{a,b}$ contains a ray of the boundary line of $P_{a,b}$, the field $\{e(n, m)\}$ has no column-to-column innovations with respect to any *other* RNSHP order induced by $(\alpha, \beta) \neq (a, b)$ (this was implicit in [8]).

It turns out that the analysis of the spectral and asymptotic properties of the field (1) does not require $\{x(n)\}$ to be purely-indeterministic. Let x_u and x_v denote the purely-indeterministic and deterministic parts in the Wold decomposition of x, respectively.

Theorem 2.3. *The evanescent field $x_u(na + mb)\lambda^{nc+md}$, defined by the purely-indeterministic part of the one-dimensional Wold decomposition of $\{x(n)\}$, is precisely the evanescent part of $\{e(n, m)\}$, (all with respect to the order induced by a and b).*

Proof. By the representation of $\{e(n,m)\}$ (see also the proof of Proposition 2.1) we have

$$\mathcal{H} = c.l.m.\{e(n,m) : (n,m) \in \mathbb{Z}^2\} = c.l.m.\{x(n) : n \in \mathbb{Z}\}$$

Denote the remote pasts $\mathcal{X}_{-\infty} = \cap_{n\in\mathbb{Z}}\mathcal{X}_n$ and $\mathcal{E}_{-\infty} = \cap_{n,m\in\mathbb{Z}}\mathcal{E}_{n,m}$. The orthogonal sum $e(n,m) = x_u(na+mb)\lambda^{nc+md} + x_v(na+mb)\lambda^{nc+md}$ is deterministic, and $\{x_u(na+mb)\lambda^{nc+md}\}$ is evanescent as was shown in the proof of Proposition 2.1. Since $\{x_v\}$ is deterministic, $\{x_v(na+mb)\lambda^{nc+md}\}$ has no column-to-column innovations with respect to any RNSHP order, i.e., it is a *half plane deterministic* field, [8]. Furthermore, since $\{x_v\}$ is deterministic, $\{x_v(n) : n \in \mathbb{Z}\} \subset \mathcal{X}_{-\infty}$. Hence $\{x_v(na+mb)\lambda^{nc+md} : (n,m) \in \mathbb{Z}^2\} \subset \mathcal{X}_{-\infty}$. Let $e = e_1 + e_2$ be the two-dimensional Wold decomposition of e [16], where the orthogonal e_1 and e_2 are the evanescent and the remote past parts, respectively. (The field e is deterministic and admits no innovation part). Since e_2 is the remote past part, $\{e_2(n,m) : (n,m) \in \mathbb{Z}^2\} \subset \mathcal{E}_{-\infty}$. In the proof of Proposition 2.1 it was shown that $\mathcal{E}_{-\infty} = \mathcal{X}_{-\infty}$, therefore $\{e_2(n,m)\}$ and $\{x_v(na+mb)\lambda^{nc+md}\}$ belong to the same subspace, and $\{e_1(n,m)\}$ and $\{x_u(na+mb)\lambda^{nc+md}\}$ belong to the same orthogonal complement. Since $e(n,m) = e_1(n,m)+e_2(n,m) = x_u(na+mb)\lambda^{nc+md}+x_v(na+mb)\lambda^{nc+md}$, the uniqueness of the orthogonal sum yields that $e_1(n,m) = x_u(na+mb)\lambda^{nc+md}$ and $e_2(n,m) = x_v(na+mb)\lambda^{nc+md}$. □

Since the case where $ad - bc = -1$ simply amounts to interchanging the roles of "past" and "future" in the lexicographic order definition that corresponds to the case where $ad - bc = 1$, we assume throughout this paper that $ad - bc = 1$. The analysis of the alternative case is identical. Since $ad - bc = 1$ we have $g.c.d.(a,b) = g.c.d.(c,d) = 1$ (see §4). Along the analysis, the order parameters a and b which determine the slope of the RNSHP, are assumed fixed.

Suppose a given random field $\{e(n,m)\}$ has a representation (1), with our fixed a and b. This representation seems to be uniquely determined only up to c and d (which exist since a and b are coprime). More precisely, let $c' \neq c, d' \neq d$ be two other integers satisfying $ad' - bc' = 1$. Since c, d and c', d' are different solutions of the same linear Diophantine equation $ax - by = 1$, there exists an integer t such that $c = c' + ta$ and $d = d' + tb$. Hence, we have from (1) that $e(n,m) = x(na+mb)\lambda^{nc+md} = x(na+mb)\lambda^{t(na+mb)}\lambda^{nc'+md'}$. Thus, different choices of c and d are interpreted as rotations of $x(k)$, by $(\lambda^t)^k$, for a suitable t. Nevertheless, by employing geometrical considerations we show in §4 that for any given field $\{e(n,m)\}$, the parameters c and d are *unique*.

3. The spectral representation

Consider the random field defined in (1). Recall that $\mathcal{H}$ is the closed subspace generated by $\{x(n) : n \in \mathbb{Z}\}$, and U is the unitary operator on $\mathcal{H}$ satisfying $x(n) = U^n x(0)$. The operator $V = \lambda I$ acts on $\mathcal{H}$ (I is the identity on $\mathcal{H}$) and defines a unitary representation of $\mathbb{Z}$ which commutes with U. Clearly, $\lambda^n = \int z^n d\delta_\lambda(z)$ where δ_λ is the Dirac measure on Γ concentrated at λ (the usual Lebesgue integral). Note that $\delta_\lambda I$ is the spectral family associated to V and $V^m = \int z^m d\delta_\lambda(z) I$. Clearly,

$e(n, m) = U^{na+mb}V^{nc+md}x(0)$. In this section we find the spectral representation of $\{U^n V^m\}$, and hence of e.

Let $(\mathbf{X}, \Sigma)$ be a measurable space. Let μ and ν be two finite measures defined on this space. We consider the product measure space $(\mathbf{X} \times \mathbf{X}, \Sigma \otimes \Sigma, \mu \times \nu)$. Let $T_1, T_2 : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ be measurable transformations. Then the transformation $T = (T_1, T_2)$ is a measurable transformation from $\mathbf{X} \times \mathbf{X}$ to itself since it has measurable sections. Assume we have a $\Sigma \otimes \Sigma$ (Borel) measurable function f then in this notation we have

$$\int_{\mathbf{X}\times\mathbf{X}} f \circ T d(\mu \times \nu) = \int_{\mathbf{X}\times\mathbf{X}} f d\left[(\mu \times \nu) \circ T^{-1}\right] \tag{4}$$

It is easy to prove the equality above by the usual procedure, starting with simple functions (see e.g., Halmos [12]).

In the following the integration space is omitted from the integral notation whenever integration is performed on the entire space. Denote $\mathbf{X}^2 = \mathbf{X} \times \mathbf{X}$. Let $A \times B \in \Sigma \otimes \Sigma$ be a measurable rectangle; then

$$\begin{aligned} T^{-1}(A \times B) &= \{(x, y) \in \mathbf{X}^2 \ : T_1(x, y) \in A, T_2(x, y) \in B\} \\ &= \{(x, y) \in \mathbf{X}^2 \ : (x, y) \in T_1^{-1}(A), (x, y) \in T_2^{-1}(B)\} = T_1^{-1}(A) \cap T_2^{-1}(B) \end{aligned}$$

Let E be the projection valued spectral measure of U. We want to define the product $E \times \delta_\lambda I$ on Γ^2. Each factor is a given algebra homomorphism from the Boolean algebra of Borel sets of Γ to the algebra of orthogonal projections in $\mathcal{H}$. Since U and V commute, so do E and δ_λ; hence, for any measurable rectangle $A \times B$, $E(A)\delta_\lambda(B)$ is an orthogonal projection. The σ-additivity on the algebra of measurable rectangles of Γ^2 of the operator valued function $E \times \delta_\lambda I$ is due to Riesz and Nagy [22, §111]. The fact that it preserves the product (intersection) in that algebra easily follows from the fact that E and δ_λ do. Now, we can define

$$\int_\Gamma \int_\Gamma P(z_1, z_2) E(dz_1) \times \delta_\lambda(dz_2) I$$

as a Stieltjes-Riemann integral for any trigonometric polynomial P [22], and hence for any continuous function on Γ^2. For fixed $x, y \in \mathcal{H}$ and for any continuous function $f \in C(\Gamma^2)$, the formula

$$\begin{aligned} \Phi_{x,y}(f) &= \left\langle\left[\int_\Gamma \int_\Gamma f(z_1, z_2) E(dz_1) \times \delta_\lambda(dz_2) I\right] x, y\right\rangle \\ &= \int_\Gamma \int_\Gamma f(z_1, z_2) \langle E(dz_1) \times \delta_\lambda(dz_2) I x, y\rangle \end{aligned}$$

defines a linear functional on $C(\Gamma^2)$, which is bounded since $E \times \delta_\lambda I$ is. By Riesz's theorem for bounded linear functionals on $C(\Gamma^2)$, there is a unique regular measure $\mu_{x,y}$ on the torus which represents the functional $\Phi_{x,y}$, and for any measurable rectangle $A \times B$, $\mu_{x,y}(A \times B) = \langle E(A)\delta_\lambda(B) I x, y\rangle$. Let $x = k_1 x_1 + k_2 x_2$ where $x_1, x_2 \in \mathcal{H}$ and $k_1, k_2 \in \mathbb{C}$; clearly, for any measurable rectangle D we have $\mu_{x,y}(D) = k_1 \mu_{x_1,y}(D) + k_2 \mu_{x_2,y}(D)$. Since both sides are measures, by the uniqueness of the extension theorem equality holds for any measurable set. In a similar way we see that $\mu_{x,y}(D)$ is linear in y and self adjoint. Since $\mu_{x,x}$ is a positive finite measure, $\mu_{x,x}(D)$ is a bounded quadratic form for any Borel set D. Finally, $\mu_{x,y}(D)$ is a self adjoint bilinear form. Hence, for any Borel set D, $\mu_{x,y}(D)$ defines a unique self adjoint operator, denoted by $F(D)$, so that $\mu_{x,y}(D) = \langle F(D)x, y\rangle$, and on the rectangles $F = E \times \delta_\lambda I$. In a similar way, it can be shown that for any Borel set D, $F(D)$ is an idempotent, i.e., it is an orthogonal projection. For any $x, y \in \mathcal{H}$, $\mu_{x,y}$ is a σ-additive measure, hence, $F(\cdot)$ is a weakly σ-additive projection measure. For any orthogonal projection measure, weak σ-additivity is equivalent to σ-additivity in the strong operator topology Dunford-Schwartz [6], and we define $E \times \delta_\lambda I(\cdot) := F(\cdot)$. By considering two applications of the extension theorem, we see that, $[E \times \delta_\lambda I](D)\,[E \times \delta_\lambda I](G) = [E \times \delta_\lambda I](D \cap G)$ for any measurable sets D, G: in the first, we see that it is true for any rectangle D and any measurable set G, and then we see that it is true for any pair of measurable sets. Hence, $E \times \delta_\lambda I$ preserves the product. Summarizing what we have already proved

Theorem 3.1. *There is a unique resolution of the identity for $\{U^n V^m\}$, namely, $E \times \delta_\lambda I$ and we have,*

$$U^n V^m = \int_\Gamma \int_\Gamma z_1^n z_2^m d(E(z_1) \times \delta_\lambda(z_2) I)$$

where $E \times \delta_\lambda I$ is an algebra homomorphism from the Boolean algebra of Borel sets of the torus to the algebra of orthogonal projections, and it is a strongly σ-additive operator valued measure.

Clearly, using similar techniques, for any $x \in \mathcal{H}$, $(E \times \delta_\lambda I)x = Ex \times \delta_\lambda$. Since $w(\cdot) = E(\cdot)x(0)$, $(E \times \delta_\lambda I)x(0) = w \times \delta_\lambda$.

Theorem 3.2. *The spectral representation of e has the form*

$$e(n,m) = \int_\Gamma \int_\Gamma z_1^n z_2^m d(w(z_1^d z_2^{-c}) \times \delta_\lambda(z_1^{-b} z_2^a)) \tag{5}$$

Proof. By the equality $e(n,m) = U^{na+mb} V^{nc+md} x(0)$, we apply Theorem 3.1 to obtain,

$$\begin{aligned} e(n,m) &= \int \int z_1^{na+mb} z_2^{nc+md} d\,[E(z_1) \times \delta_\lambda(z_2) I]\, x(0) \\ &= \int \int (z_1^a z_2^c)^n (z_1^b z_2^d)^m d\,[w(z_1) \times \delta_\lambda(z_2)] \end{aligned}$$

Define $T_1(z_1, z_2) = z_1^a z_2^c$ and $T_2(z_1, z_2) = z_1^b z_2^d$; then the transformation $T = (T_1, T_2)$ from the torus $\Gamma \times \Gamma$ to itself is measurable. As we saw, in order to evaluate the integral above we need to know how the transformation T^{-1} acts. Due to (4), the unique intersection point can be used to calculate the inverse, and we obtain

$$T^{-1}(\gamma_1, \gamma_2) = \left(\gamma_1^d \gamma_2^{-c}, \gamma_1^{-b} \gamma_2^a\right) \tag{6}$$

(Clearly, the invertibility of T follows from the constraint $ad - bc = 1$). By considering functionals, (4) holds for vector measures, and by applying it to the function $f(z_1, z_2) = z_1^n z_2^m$, which is measurable (in fact continuous) the result follows. □

Remark. T is an automorphism of Γ^2 with the usual operation. In fact any continuous automorphism of Γ^2 has the form $(x, y) \mapsto (x^a y^b, x^c y^d)$ where a, b, c, and d are integers, and $| ad - bc |= 1$ see [13].

The problem has a representation in the $[0, 2\pi]^2$ space by the usual homomorphism $\chi : t \mapsto e^{it}$. Of course, we should consider the suitable homomorphism on the torus. The transformations corresponding to T_1 and T_2 are non-parallel straight lines; because of folding, each line modulo 2π splits into a finite number of parallel lines which intersect the axis (ω and ν) in a periodic way; in fact, the line corresponds to T_1 intersects the ω-axis a times and the ν-axis c times, and the line corresponds to T_2 intersects the ω-axis b times and the ν-axis d times. Thus, in the following when the intersection point of the two lines corresponding to T_1 and T_2 is considered, it is understood that we refer to the original unique intersection point in the entire plane, whose coordinates are evaluated modulo 2π. Following this convention there exists only a single intersection point in the $[0, 2\pi]^2$ space.

Theorem 3.3. *In $[0, 2\pi]^2$, the spectral representation of e has the form*

$$e(n, m) = \int_{[0,2\pi]^2} e^{i(n\omega + m\nu)} d(\tilde{w}(\omega d - \nu c) \times \tilde{\delta}_\rho(\nu a - \omega b)) \tag{7}$$

where $\rho = \text{Arg}(\lambda)$ and the measures $\tilde{w}$ and $\tilde{\delta}_\rho$ are the measures on $[0, 2\pi]$ corresponding to w and δ_λ by the homomorphism χ.

It is clear that the stochastic spectral measure of $e(n, m)$ is concentrated on a straight line with rational slope, corresponding to a and b.

Following the steps of the construction employed in the proof of Theorem 3.2 (and Theorem 3.3) in reverse order and according to the discussion in the introduction, we obtain the converse theorem:

Theorem 3.4. *Let $\{x(n)\}$ be a weakly stationary process with stochastic spectral measure $\tilde{w}$. If $\{e'(n, m)\}$ is a homogeneous random field with stochastic spectral measure concentrated on a straight line with slope parameter (a, b), and e' has the form (7) with parameters (c, d), then $e'(n, m) = x(na + mb)\lambda^{nc+md}$.*

The representation above also yields a spectral representation for the covariance sequence. Let μ be the spectral measure of $x(0)$; clearly on measurable rectangles we have $|| \ (E \times \delta_\lambda I)x(0) \ ||^2 = \mu \times \delta_\lambda$. The two sides are measures, and from the uniqueness of the extension we have equality for every Borel set. The spectral measure $\mu =|| \ Ex(0) \ ||^2$ is also called the "spectral measure of the covariances of $\{x(n)\}$" since $\langle x(n), x(0)\rangle = \int z^n d\mu(z)$.

Theorem 3.5. *Denote the covariance sequence of $e(n, m)$ by $R(n, m)$; then*

$$R(n, m) = \int_\Gamma \int_\Gamma z_1^n z_2^m d(\mu(z_1^d z_2^{-c}) \times \delta_\lambda(z_1^{-b} z_2^a)) \tag{8}$$

where μ is the spectral measure corresponding to $\{x(n)\}$.

In the $[0, 2\pi]^2$ space we have

Theorem 3.6. *With the same notation as in Theorem 3.5*

$$R(n,m) = \int_{[0,2\pi]^2} e^{i(n\omega+m\nu)} d(\tilde{\mu}(\omega d - \nu c) \times \tilde{\delta}_\rho(\nu a - \omega b)) \tag{9}$$

where $\rho = \text{Arg}(\lambda)$ *and the measures* $\tilde{\mu}$ *and* $\tilde{\delta_\rho}$ *are the measures on* $[0, 2\pi]$ *corresponding to* μ *and* δ_λ *by the homomorphism* χ.

The proof easily follows from Theorem 3.2 and Theorem 3.3.

Again, following the steps of the construction from last to first we obtain the converse theorem. Since the covariance spectral representation determines the stochastic representation Doob [5], Grenander [11], we have

Theorem 3.7. *A two-dimensional homogenous random field has the form* $x(na + mb)\lambda^{nc+md}$ *(with* $a, b, c,$ *and* d *are integers satisfying* $ad - bc = 1$ *and* $\lambda \in \Gamma$*) if and only if it has a spectral measure concentrated on a straight line in the way described above.*

It is interesting to note here that in the case where $\{x(n)\}$ is a wide sense stationary and purely-indeterministic process, the spectral measure of $\{e(n, m)\}$, being atom-less on the line it is concentrated on, is singular continuous. Nevertheless, the spectral distribution of this field is discontinuous along this straight line.

Of course, we can extend all these discussions to the case where we have a finite sum of such orthogonal processes, i.e.,

$$e(n,m) = \sum_{k=1}^{P} x_k(na_k + mb_k)\lambda_k^{nc_k+md_k} \tag{10}$$

where $a_k d_k - b_k d_k = 1$, $\{\lambda_k\}_{k=1}^P \subset \Gamma$ and $\{x_1(n)\}$, $\{x_2(n)\} \cdots \{x_P(n)\}$ is a family of P orthogonal weakly stationary processes. By the orthogonality, it is enough to apply the theorems to each term and obtain the corresponding result. Note that in the framework of the 2-D Wold decomposition, [8], where the evanescent fields are those producing the column-to-column innovations of the deterministic component of the decomposition, all evanescent components of the decomposition are mutually orthogonal. Hence, no orthogonality assumptions are needed in extending the foregoing results.

4. The number theoretic approach

As we have already seen, the spectral measure of the process $\{e(n, m)\}$ is concentrated on a line. Hence, it is not absolutely continuous with respect to the Lebesgue measure on the torus, and therefore $\{e(n, m)\}$ has no spectral density function. In such cases, the covariance sequence is not absolutely summable, i.e., $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} | R(n,m) |= \infty$. We wish to find a summation order so that the two-dimensional trigonometric series generated by $\{R(n, m)\}$ will converge conditionally such that its limit will exhibit the behavior described in the preceding sections, i.e., that it is concentrated on a straight line.

Since we deal with straight lines on the lattice, we will need the following well known result (e.g., see Andrews [1]).

Let a and b be two non-zero integers and c some other integer. The equation

$$ax + by = c$$

is called the *linear Diophantine equation*. A *solution* of this equation is a pair (x, y) of integers (a *lattice point* in the plane) that satisfies the equation.

Theorem. *The linear Diophantine Equation*

$$ax + by = c$$

has a solution if and only if $d \mid c$ *where* $d = g.c.d.(a, b)$. *Furthermore, if* (x_0, y_0) *is a solution of this equation, then the set of solutions of the equation consists of all integer pairs* (x, y) *of the form*

$$x = x_0 + t\tfrac{b}{d} \quad \text{and} \quad y = y_0 - t\tfrac{a}{d},\ t \in \mathbb{Z} \tag{11}$$

Note: if a and b are coprime then there will always be solutions, given by (11).

Consider once again the random field (1), with $\rho = \operatorname{Arg}(\lambda)$, where as before, a, b, c, and d are integers satisfying $ad - bc = 1$, $\rho \in [0, 2\pi)$, and $\{x(n)\}$ is a weakly stationary process. Since we wish to consider trigonometric series expansions, we will restrict ourselves to processes $\{x(n)\}$ having an absolutely summable covariance sequence $\{r(n)\}$. The trigonometric series expansion of the covariance sequence of such a process is continuous (in fact uniformly continuous), and is the Radon-Nykodim derivative of the spectral measure of the process (see e.g. [5]). The covariance sequence of e is

$$R(n, m) = r(na + mb)e^{i\rho(nc+md)}$$

We wish to find a summation order Q on $\mathbb{Z}^2$ such that the sum

$$\begin{aligned} S(\omega, \nu) &= \sum_{(n,m)\in Q} R(n, m)e^{-i(\omega n+\nu m)} \\ &= \sum_{(n,m)\in Q} r(na + mb)e^{i\rho(nc+md)}e^{-i(\omega n+\nu m)} \end{aligned}$$

is defined (in some sense) and has the same analytic behavior as the corresponding spectral measure of e, obtained in §3. Hence, we will sum the series along diagonals in the lattice plane, corresponding to the solutions of a linear Diophantine equation for a slope given by $-a/b$, i.e., along the lines $na + mb = k$ for some arbitrary integer k. Since $ad - bc = 1$, by the previous theorem a and b are coprime. We have a particular solution (n_k, m_k) for each k, in fact countably many solutions which by (11) are of the form

$$\begin{aligned} n &= n_k + tb \\ m &= m_k - ta \end{aligned} \tag{12}$$

where t is an arbitrary integer. The union over k of these solutions covers all the lattice plane.

Theorem 4.1. *With the foregoing notation and with respect to the order of summation along the lines of slope $-a/b$ through $\{(n_k, m_k)\}$, we have*

$$S(\omega, \nu) = \tilde{S}(\omega, \nu) \cdot d\delta\,[\rho - (\nu a - \omega b)]$$

where $d\delta$ is the one-dimensional Fourier-Stieltjes series (with its argument given by $\rho - (\nu a - \omega a)$) of the Dirac measure δ concentrated at the origin. $\tilde{S}$ is some continuous periodic function on $[0, 2\pi]^2$.

Proof. Consider the set $\{(n_k, m_k)\}$ of particular solutions of the linear Diophantine equation $na + mb = k$, given by $n_k a + m_k b = k$ for every $k \in \mathbb{Z}$. Using (12) we define the partial sums

$$\begin{aligned} S^{K,T}(\omega, \nu) &= \sum_{k=-K}^{K} \sum_{t=-T}^{T} r(k) e^{i\rho[n_k c + m_k d - t(ad-bc)]} e^{-i[n_k \omega + m_k \nu + t(\omega b - \nu a)]} \\ &= \sum_{k=-K}^{K} r(k) e^{-i[n_k\omega + m_k \nu - \rho(n_k c + m_k d)]} \sum_{t=-T}^{T} e^{-it[\rho - (\nu a - \omega b)]} \end{aligned} \tag{13}$$

Since by assumption, the sequence $\{r(k)\}$ is absolutely summable, the first sum converges uniformly to some continuous and periodic function, that we denote by $\tilde{S}(\omega, \nu)$. The second sum is the partial sum of the Fourier-Stieltjes series of the Dirac measure δ concentrated at the origin, Zygmund [24]. Hence, in this sense the two iterated limits exist and we have

$$\lim_{K\to\infty} \lim_{T\to\infty} S^{K,T} = \lim_{T\to\infty} \lim_{K\to\infty} S^{K,T} = \lim_{K,T\to\infty} S^{K,T}.$$

We therefore define

$$S(\omega, \nu) := \lim_{K\to\infty} \lim_{T\to\infty} S^{K,T}(\omega, \nu) = \tilde{S}(\omega, \nu) \cdot d\delta\,[\rho - (\nu a - \omega b)]\,.$$

We finally note that the convergence of $S^{K,T}$ is not a pointwise convergence, as the second term does not converge pointwise at any point, and at the origin it monotonically tends to infinity. □

Remarks.

1. As implied by the last proof, S is not unique and depends on the choice of the particular solutions $\{n_k, m_k\}$.
2. We can relax the assumption of absolute summability of $\{r(k)\}$; in that case, the first sum will be related to the Fourier-Stieltjes series of the spectral measure μ, and we obtain a representation closely related to that of §3.
3. By definition, S, $\tilde{S}$, and $d\delta$ are periodic.

To achieve uniqueness of S, i.e., independence of $\{n_k, m_k\}$, we shall consider a normalized version of the double sum in (13): Let $s \in [0, 2\pi)$ (modulo 2π); since the partial sums $\sum_{t=-T}^{T} e^{ist}$ are bounded for any $s \neq 0$, the normalized sums $\frac{1}{2T} \sum_{t=-T}^{T} e^{ist}$ tend to zero. Clearly, for $s = 0$ these averages converge to 1.

We thus have, $\lim_{T\to\infty} \frac{1}{2T}\sum_{t=-T}^{T} e^{ist} = \delta(\{s\})$ where δ is the Dirac measure at the origin. Hence, we define $\sigma^{K,T}(\omega,\nu) = \frac{1}{2T}S^{K,T}(\omega,\nu)$. Based on the arguments made in the proof of Theorem 4.1 and the definition of $\sigma^{K,T}$, we conclude that the two iterated limits in K and T exist pointwise everywhere, and we denote $\sigma = \lim_{K\to\infty}\lim_{T\to\infty}\sigma^{K,T} = \lim_{K\to\infty}\lim_{T\to\infty}\frac{1}{2T}S^{K,T}$.

Theorem 4.2. *With the foregoing notation and with respect to the order of summation along the lines of slope $-a/b$ through $\{(n_k, m_k)\}$, we have*

$$\sigma(\omega,\nu) = \tilde{S}(\omega,\nu)\cdot\delta(\{\rho-(\nu a-\omega b)\})$$

where δ is the Dirac measure concentrated at the origin, and $\tilde{S}$ is some continuous periodic function on $[0,2\pi]^2$. Furthermore, the function σ is uniquely determined, independently of $\{(n_k, m_k)\}$.

Proof. The limit σ vanishes at all points not on the line $\rho = \nu a - \omega b$, since for points not on this line the second sum in (13) converges to zero when normalized by $\frac{1}{2T}$. We therefore have

$$\begin{aligned}\sigma(\omega,\nu) &= \lim_{K\to\infty}\lim_{T\to\infty}\sigma^{K,T}(\omega,\nu)\\ &= \tilde{S}(\omega,\nu)\cdot\delta(\{\rho-(\nu a-\omega b)\})\end{aligned} \tag{14}$$

It only remains to show the uniqueness of σ. Let $\{t_k\}_{k\in\mathbb{Z}}$ be an arbitrary sequence of integers, and let σ' be the function obtained by (14) and the method as in the proof of Theorem 4.1 with the choice of the particular solutions given by

$$n_k' = n_k + t_k b,\quad m_k' = m_k - t_k a \tag{15}$$

From (13) we conclude that it is only $\tilde{S}$ that is affected by changing the set of particular solutions. Let $\tilde{S}'$ be the function associated with $\{(n_k', m_k')\}$; clearly, we need to show that

$$\begin{aligned}\sigma(\omega,\nu)-\sigma'(\omega,\nu) &= \left[\tilde{S}(\omega,\nu)-\tilde{S}'(\omega,\nu)\right]\delta(\{\rho-(\nu a-\omega b)\})\\ &\equiv 0\end{aligned}$$

In other words, we only need to show that $\tilde{S}(\omega,\nu)-\tilde{S}'(\omega,\nu)\equiv 0$ on the line $\rho-(\nu a-\omega b)=0$. Using (13) and (15),

$$\begin{aligned}&\tilde{S}(\omega,\nu)-\tilde{S}'(\omega,\nu)\\ &\quad=\sum_{k\in\mathbb{Z}} r(k)e^{-i[n_k\omega+m_k\nu-\rho(n_kc+m_kd)]}-\sum_{k\in\mathbb{Z}} r(k)e^{-i\left[n_k'\omega+m_k'\nu-\rho(n_k'c+m_k'd)\right]}\\ &\quad=\sum_{k\in\mathbb{Z}} r(k)e^{-i[n_k\omega+m_k\nu-\rho(n_kc+m_kd)]}\{1-e^{it_k[\rho-(\nu a-\omega b)]}\}\end{aligned}$$

and on the above line the second factor is identically zero. □

Of course, if we have a finite sum, as in (10), of orthogonal processes of the form (1), it is enough to apply the same procedure to each term.

Although the deterministic field e has no spectral density function, for convenience we may think of $\sigma(\omega, \nu)$ as its spectral density, and call σ *"the spectral pseudo-density"*. Theorem 4.3 bellow justifies this name. Theorem 4.2 asserts that the spectral pseudo-density function is non-zero on the same straight line as in §3. It remains to find the relation between μ (the spectral measure of $\{x(n\}$) and $\tilde{S}$. We would like to develop an expression that can provide us with an interpretation of the geometric properties of this spectral pseudo-density. In the frequency plane, we are interested only in the points on the "pseudo-density support line" $\rho = \nu a - \omega b$.

Consider the intersection point of the line $\omega d - \nu c = 0$ with the pseudo-density support line, and denote it by $(\omega_0, \nu_0) = (\rho c, \rho d)$. Fix this point as our new origin, and consider the pseudo-density support line as our one-dimensional axis system. The point $+h$ corresponds to the point $(\omega_0 + \frac{ah}{r}, \nu_0 + \frac{bh}{r})$, in the (ω, ν) plane, while the point $-h$ corresponds to the point $(\omega_0 - \frac{ah}{r}, \nu_0 - \frac{bh}{r})$, where $r = \sqrt{a^2 + b^2}$. In the new coordinate system we have $\sigma(\omega(h), \nu(h)) = \tilde{S}(\omega(h), \nu(h))$, where $(\omega(h), \nu(h))$ is a point on the pseudo-density support line with distance $+h$ from the "origin" (ω_0, ν_0).

Theorem 4.3. *Let* $\Psi(h) := \tilde{S}(\omega(h), \nu(h))$. *Then*

$$\Psi(\pm h) = s_x(\frac{\pm h}{\sqrt{a^2 + b^2}})$$

where s_x *is the spectral density of* $\{x(k)\}$. *Furthermore, we have* $s_x = \frac{d\mu}{dm}$, *the Radon-Nykodim derivative of the spectral measure* μ *of* $\{x(k)\}$ *with respect to the Lebesgue measure* m.

Proof. Since we assumed that $\{r(k)\}$ is absolutely summable, $\sum_k r(k)e^{-ik\theta}$ is well defined. As we mentioned at the beginning of the section, this sum converges to $s_x(\theta) = \frac{d\mu}{dm}(\theta)$, which is the spectral density of $\{x(k)\}$. Now, by the definition of $\tilde{S}$, as given in the proof of Theorem 4.1, we have

$$\begin{aligned}
\Psi(h) &= \sum_k r(k)e^{-i[n_k\omega(h)+m_k\nu(h)-\rho(n_kc+m_kd)]} \\
&= \sum_k r(k)e^{-i\left[n_k(\rho c+\frac{ah}{\sqrt{a^2+b^2}})+m_k(\rho d+\frac{bh}{\sqrt{a^2+b^2}})-\rho(n_kc+m_kd)\right]} \\
&= \sum_k r(k)e^{-i(k\frac{h}{\sqrt{a^2+b^2}})} \\
&= s_x(\frac{h}{\sqrt{a^2+b^2}})
\end{aligned}$$

In a similar way for the negative ray of the pseudo-density support line

$$\Psi(-h) = s_x(-\frac{h}{\sqrt{a^2 + b^2}}).$$

Now, the relation between $\tilde{S}$ and μ is clear and the result follows. □

Remarks.

1. The field e is associated with a distinguished point, that we name the *middle point*. The middle point is a point on the pseudo-density support line, and is in fact the origin of the 1-D axis defined on this support line.

2. On the line supporting the spectral pseudo-density, the pseudo-density function depends only on the spectral density of the covariance sequence $\{r(k)\}$, while the parameters a, b contribute only to scaling.

3. The above procedure can be extended to cases where there is a finite sum of orthogonal processes. In that case, each process has its middle point and similar representation exists on the line supporting each of the pseudo-spectral densities.

4. Note that since $s_x(\theta)$ is the spectral density of $\{x(k)\}$, $s_x(-\theta) = s_x^*(\theta)$. As θ is measured relative to the middle point, and since the middle point coordinates are a unique function of c and d, there is only a *single* pair c, d that satisfies the symmetry properties of $s_x(\theta)$.

5. Asymptotic properties

In this section we investigate the asymptotic properties of the processes introduced in the previous sections. The classical ergodic theorems (see e.g. [5]) assert that a necessary and sufficient condition for a one-dimensional weakly stationary process to be mean ergodic is that its spectral measure has no atom (i.e., is continuous) at the origin. For a circular Gaussian (i.e., its real and imaginary parts are independent and have identical Gaussian distributions) one-dimensional weakly stationary process, a necessary and sufficient condition for mean ergodicity of its second order moment is that its spectral measure has no atoms (an application of Theorem 7.1 in [5, p. 493]). We would like to find necessary and sufficient conditions for the field (1) to be mean ergodic in the first and the second order moments.

Proposition 5.1. *A necessary and sufficient condition for a two-dimensional weakly stationary complex field to be mean ergodic is that its spectral measure has no atom at the origin.*

Proof. Let $\{y(n,m)\}$ be a weakly stationary field with a two-dimensional spectral measure η. Then

$$E\left|\frac{\sum_{n=1}^{N}\sum_{m=1}^{M} y(n,m)}{NM}\right|^2 = \frac{\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{p=1}^{N}\sum_{q=1}^{M} R(n-p, m-q)}{(MN)^2}$$

$$= \int_{[0,2\pi]^2} \frac{1}{(NM)^2} \frac{\sin^2 \frac{N\omega}{2}}{\sin^2 \frac{\omega}{2}} \frac{\sin^2 \frac{M\nu}{2}}{\sin^2 \frac{\nu}{2}} d\eta(\omega,\nu)$$

The integrand is uniformly bounded (less than one) and tends to zero uniformly on any neighborhood not containing the origin. In fact,

$$\lim_{N\to\infty} \frac{\sin^2 \frac{N\omega}{2}}{N^2 \sin^2 \frac{\omega}{2}} = \begin{cases} 1\ , & \omega = 0 \\ 0\ , & \omega \neq 0 \end{cases}$$

We therefore conclude using Lebesgue's bounded convergence theorem that

$$\lim_{N,M\to\infty} E \left| \frac{\sum_{n=1}^{N} \sum_{m=1}^{M} y(n,m)}{NM} \right|^2 = \eta(\{0\},\{0\})$$

where the limit exists in an unrestricted way, and the result follows. □

Applying the results of Theorem 3.6 and Proposition 5.1 to the field (1), there are two different cases where there is no atom at the origin:

Corollary 5.2. *A necessary and sufficient condition for a field $\{e(n,m)\}$ of the form (1) to be mean ergodic is that $\rho \neq 0$ or that the spectral measure of $\{x(n)\}$ has no mass at the origin, i.e., $\mu(\{0\}) = 0$. Equivalently, $\rho \neq 0$ or $\{x(n)\}$ is mean ergodic.*

Proposition 5.3. *A necessary and sufficient condition for mean ergodicity of the second order moment of a two-dimensional Gaussian circular field is that its spectral measure has no atoms.*

Proof. We assume that the field $\{y(n,m)\}$ is Gaussian and circular with covariance $\{R(k,l)\}$ and spectral measure η. In the following we wish to find conditions so that for any $k, l \in \mathbb{Z}$,

$$\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} y(n+k, m+l)\overline{y}(n,m) \to R(k,l) \tag{16}$$

in the mean as $N, M \to \infty$. Using the spectral representation of the covariances, $R(k,l) = \int e^{i(k\omega+l\nu)} d\eta(\omega,\nu)$, the circularity of the Gaussian field, and the properties of its fourth order cumulants, we have

$$E\left| \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} y(n+k, m+l)\overline{y}(n,m) - R(k,l) \right|^2$$
$$= \frac{\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{p=1}^{N}\sum_{q=1}^{M} \mid R(n-p, m-q) \mid^2}{(NM)^2}$$
$$= \int_{[0,2\pi)^2}\int_{[0,2\pi)^2} \frac{1}{(NM)^2} \frac{\sin^2 \frac{N(\omega-\omega')}{2}}{\sin^2 \frac{(\omega-\omega')}{2}} \frac{\sin^2 \frac{M(\nu-\nu')}{2}}{\sin^2 \frac{(\nu-\nu')}{2}} d(\eta(\omega,\nu)\times\eta(\omega',\nu'))$$

Using Lebesgue's bounded convergence theorem, and following similar arguments to those employed in the proof of Wiener's lemma (see, e.g., [24]) we conclude that the last expression tends to zero if and only if η is atom free, i.e., it is continuous. □

Assume that $\{x(n)\}$ is circular Gaussian is in fact equivalent to the assumption that $\{e(n, m)\}$ is circular Gaussian. Hence, for ergodicity in the mean of the second order moment of a circular Gaussian field of the form (1), it is necessary and sufficient that there are no atoms on the line $\nu a - \omega b = \rho$. Even simpler, it is necessary and sufficient to verify that μ (as a one dimensional measure) has no atoms. We thus have

Corollary 5.4. *A necessary and sufficient condition for mean ergodicity of the second order moments of a circular Gaussian field $\{e(n, m)\}$ of the form (1) is that μ has no atoms, or equivalently, that the process $\{x(n)\}$ is mean ergodic in the second order moments.*

Corollary 5.5. *A circular Gaussian evanescent field of the form (1) is mean ergodic in the first and second order moments.*

Now we consider the same problem as in the previous discussion, except that we would like the arithmetic averages to converge to the "correct" limit *almost everywhere*; in fact we are interested in the strong law of large numbers. Almost everywhere convergence of the arithmetic averages can not be achieved by an unrestricted tendency to infinity of the size of the observed sets. In the following we assume that N and M tend to infinity in a restricted way, where we assume that N and M depend on one parameter k such that $\min\{N(k), M(k)\} \to \infty$ as $k \to \infty$ and $0 < \lim_{k\to\infty} N(k)/M(k) < \infty$. Denote by $|| \bullet ||$ the Euclidian norm. Almost everywhere convergence to the correct limit will be called *strong ergodicity*.

Before introducing the main result of this section, let us consider the following special case.

Example. Let $\{x(n)\}$ be a centered weakly stationary circular Gaussian process. Hence, $\{x(n)\}$ is also stationary in the strict sense. Since $\{x(n)\}$ is circular Gaussian and stationary, $\{e(n, m)\}$ is strictly stationary as well. By the pointwise ergodic theorem, the averages converge almost everywhere (see e.g., Halmos [13], Krengel [19]). The limit is almost everywhere the limit in the mean and that is zero if and only if the spectral measure of $\{e(n, m)\}$ has no atom at the origin, i.e., $\rho \neq 0$, or $\rho = 0$ and the spectral measure of $\{x(n)\}$ has no atom at the origin. The latter condition holds if and only if $\{x(n)\}$ satisfies the strong law of large numbers.

The next theorem extends the scope of the foregoing conclusion to a much broader setting than that of Gaussian fields, using the results of Gaposhkin [10]. More specifically, we apply Theorem 3.3 of [10], to the case of homogeneous random fields, and specialize the general result to the case where the homogeneous field is given by (1).

Theorem 5.6. *Assume the spectral measure of $\{e(n,m)\}$ has no atom at the origin. Then, a necessary and sufficient condition for strong ergodicity is that*

$$\lim_{n\to\infty} \int_{0<||(\omega,\nu)||\le 2^{-n}} d(w(\omega d - \nu c) \times \delta_\rho(\nu a - \omega b)) = 0 \;\; a.e.$$

If $\rho \neq 0$, then it is clear that for a sufficiently large n the above integral vanishes everywhere. If $\rho = 0$ the condition is reduced to the condition

$$\lim_{n\to\infty} \int_{0<|\omega|\le 2^{-n}} dw(\omega) = 0 \;\; \text{a.e.}$$

which is equivalent (see [10]) to strong ergodicity of $\{x(n)\}$.

Hence for the field (1) we have

Corollary 5.7. *A necessary and sufficient condition for strong ergodicity of a field $\{e(n,m)\}$ of the form (1) is that $\rho \neq 0$; or if $\rho = 0$, that $\{x(n)\}$ satisfies the strong law of large numbers.*

Since the automorphism which generates a strictly stationary process is multiplicative [5], for fixed k, l the field $\{y(n+k, m+l)\bar{y}(n,m)\}$ is strictly stationary if $\{y(n,m)\}$ is; in that case, by the pointwise ergodic theorem the limit in (16) exist almost everywhere. Using Theorem 5.4 we have

Corollary 5.8. *A necessary and sufficient condition for strong ergodicity in the second order moments of a circular Gaussian field $\{e(n,m)\}$ of the form (1) is that the spectral measure of $\{e(n,m)\}$ has no atoms; that is the spectral measure of $\{x(n)\}$ has no atoms.*

6. The real valued case

Let $\{x^{(1)}(n)\}, \{x^{(2)}(n)\} \subset L_2$ be two orthogonal real valued weakly stationary processes; then $x(n) = x^{(1)}(n) - ix^{(2)}(n)$ is a complex stationary process (by orthogonality). In analogy to the complex random field (1), we define a real field by

$$\tilde{e}(n,m) = \Re\{x(na+mb)\lambda^{nc+md}\}$$

Writing $\tilde{e}(n,m)$ explicitly, with $\rho = \mathrm{Arg}(\lambda)$, we obtain

$$\begin{aligned} \tilde{e}(n,m) &= \Re\{[x^{(1)}(na+mb) - ix^{(2)}(na+mb)]\lambda^{nc+md}\} \\ &= x^{(1)}(na+mb)\cos[(nc+md)\rho] \\ &\quad + x^{(2)}(na+mb)\sin[(nc+md)\rho]. \end{aligned} \tag{17}$$

Beside stationarity and orthogonality of $x^{(1)}$ and $x^{(2)}$, we assume throughout this section that $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$ have the same covariance sequence (i.e., $\langle x^{(1)}(n), x^{(1)}(0)\rangle = \langle x^{(2)}(n), x^{(2)}(0)\rangle$ for every n), which is sufficient for $\tilde{e}$ to be stationary.

In the following, unless specifically stated otherwise, all the subspaces considered, spanned by the samples of $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$, are over the *real* field.

In order to consider the spectral measure of a real stationary process, we consider a larger Hilbert space, over the *complex* field, which isometrically (over the reals) contains the original one. More precisely, let $\{y(n)\}$ be a real stationary process, and define $\mathcal{Y}_\infty = c.l.m.\{y(n) : n \in \mathbb{Z}\}$. Let U be the unitary operator on $\mathcal{Y}_\infty$ associated with y, i.e., such that $U^n y(0) = y(n)$. We define the complexification of $\mathcal{Y}_\infty$ by $\bar{\mathcal{Y}}_\infty = \mathcal{Y}_\infty + i\mathcal{Y}_\infty$. Let $f, g \in \mathcal{Y}_\infty$; on $\bar{\mathcal{Y}}_\infty$ we define $\bar{U}(f+ig) = Uf + iUg$. Since U is unitary, it is easy to check that $\bar{U}$ is unitary as well, and satisfies $\bar{U}|_{\mathcal{Y}_\infty} = U$. By construction $\langle U^n y(0), y(0)\rangle = \langle \bar{U}^n y(0), y(0)\rangle$; the last sequence is positive semidefinite, and is therefore the Fourier transform of a positive measure on the unit circle Γ, the *spectral measure* of $\{y(n)\}$.

The assumption that $x^{(1)}(n)$ and $x^{(2)}(n)$ have the same covariance sequence means that they have the same spectral measure, denoted by μ, which for every n satisfies

$$\langle x^{(1)}(n), x^{(1)}(0)\rangle = \int_\Gamma \lambda^n d\mu = \langle x^{(2)}(n), x^{(2)}(0)\rangle.$$

In order to analyze $\tilde{e}$ in analogy to the analysis of e in the previous sections, we look at the Wold decomposition in Hilbert spaces over the reals.

Let $\{y(n)\}$ be a weakly stationary real process with associated unitary operator U, and define (over the reals) $\mathcal{Y}_n = c.l.m.\{y(k) : k \le n\}$. Let $\hat{y}(n)$ be the orthogonal projection of $y(n)$ on $\mathcal{Y}_{n-1}$. In the complex case, using spectral theory [5], it is easy to check that $\hat{y}(n+1) = U\hat{y}(n)$. Since we deal with real spaces, we can not apply spectral theory, so we give another proof which is purely geometric (and applies to the real or complex case); see also Hanner [14, p. 163]. Indeed, by the definition of $\hat{y}(n)$ we have $\langle y(n) - \hat{y}(n), y(k)\rangle = 0$ for all $k \le n-1$. Since U is unitary, $\langle y(n+1) - U\hat{y}(n), y(k+1)\rangle = 0$; hence $y(n+1) - U\hat{y}(n) \perp \mathcal{Y}_n$. Since $U\hat{y}(n) \in \mathcal{Y}_n$, we have $\hat{y}(n+1) = U\hat{y}(n)$; this also yields that $\|y(n) - \hat{y}(n)\|$ is independent of n. We are now in position to define the Wold decomposition as in [5, p. 571]. By the fact $\hat{y}(n+1) = U\hat{y}(n)$, it is easy to show that the purely-indeterministic part $\{y_u(n)\}$ and the deterministic part $\{y_v(n)\}$ satisfy $y_u(n) = U^n y_u(0)$ and $y_v(n) = U^n y_v(0)$. Considering the complexification of the original space, the Wold decomposition of $\{y(n)\}$ yields the same spectral decomposition as in the complex case.

We now compare the Wold decompositions of a real stationary process $\{y(n)\}$, obtained over the reals in the real L_2 (denoted in this discussion by $L_2^{(\mathbb{R})}$) and over $\mathbb{C}$ in the complex L_2. As noted in the discussion above, $\{y(n)\}$ remains weakly stationary in L_2, with the same spectral measure. Define $\bar{\mathcal{Y}}_\infty = \mathcal{Y}_\infty + i\mathcal{Y}_\infty$ and $\bar{\mathcal{Y}}_n = \mathcal{Y}_n + i\mathcal{Y}_n$ (recall that $\mathcal{Y}_\infty$ and $\mathcal{Y}_n$ are defined over the reals, in $L_2^{(\mathbb{R})}$), and let $\bar{y}(n)$ and $\hat{y}(n)$ be the orthogonal projections (each over the corresponding field) of $y(n)$ on $\bar{\mathcal{Y}}_{n-1}$ and on $\mathcal{Y}_{n-1}$, respectively. Then $\hat{y}(n) \in \mathcal{Y}_{n-1} \subset \bar{\mathcal{Y}}_{n-1}$, and for every $k < n$ we have $\langle y(n) - \hat{y}(n), y(k)\rangle = 0$, which implies $y(n) - \hat{y}(n) \perp \bar{\mathcal{Y}}_{n-1}$. The uniqueness of the orthogonal projection yields $\bar{y}(n) = \hat{y}(n)$.

Lemma 6.1. *A real weakly stationary process has the same Wold decomposition over the complex field or over the reals.*

Proof. The construction of the Wold decomposition is by looking at the orthogonal projections of $y(n)$ onto:

(i) the closed subspace generated over $\mathbb{R}$ by $\{y(k) - \hat{y}(k) : k \in \mathbb{Z}\}$ in the real case,
(ii) the closed subspace generated over $\mathbb{C}$ by $\{y(k) - \bar{y}(k) : k \in \mathbb{Z}\}$ in the complex case.

Since the generators, $\{y(k) - \bar{y}(k) : k \in \mathbb{Z}\}$, are the same and are *real* functions, we have that $\langle y(n), y(k) - \bar{y}(k)\rangle$ in the complex L_2 is real for every k and n, so the projections are the same (though the spaces are not!). □

Let $\{x(n)\}$ and $\{y(n)\}$ be two weakly stationary processes (real or complex), and define the following subspaces (over the relevant field). $\mathcal{X}_n = c.l.m.\{x(k) : k \le n\}$ $\mathcal{X}_\infty = c.l.m.\{x(n) : n \in \mathbb{Z}\}$, and analogously define $\mathcal{Y}_n$, $\mathcal{Y}_\infty$ for $\{y(n)\}$. Let $\hat{x}(n)$ be the orthogonal projection of $x(n)$ on $\mathcal{X}_{n-1}$, and define $\hat{y}(n)$ similarly.

Proposition 6.2. *Let $\{x(n)\}$ and $\{y(n)\}$ be orthogonal weakly stationary processes, with the same spectral measure. Define $\xi(n) = \frac{x(n) - \hat{x}(n)}{\sigma}$, with $\sigma = \|x(0) - \hat{x}(0)\|$, and let $c_j := \langle x(0), \xi(-j)\rangle$, and $\eta(n) := \frac{y(n) - \hat{y}(n)}{\sigma}$. If $z(n) := x(n) + y(n)$, then*

(i) $\{z(n)\}$ is weakly stationary, and $\hat{z}(n) = \hat{x}(n) + \hat{y}(n)$.
(ii) $z_u = x_u + y_u$ and $z_v = x_v + y_v$, where x_u, y_u, z_u and x_v, y_v, z_v, are the purely-indeterministic and deterministic components in the Wold decomposition of x, y, z, respectively.
(iii) $x_u(n) = \sum_{j=0}^{\infty} c_j \xi(n-j)$ and $y_u(n) = \sum_{j=0}^{\infty} c_j \eta(n-j)$, hence

$$z_u(n) = \sum_{j=0}^{\infty} c_j [\xi(n-j) + \eta(n-j)].$$

Proof. (i) Let U_1, U_2 be the unitary operators associated with $\{x(n)\}$, $\{y(n)\}$, respectively. On the direct sum $\mathcal{X}_\infty \oplus \mathcal{Y}_\infty = c.l.m.\{x(n), y(n) : n \in \mathbb{Z}\}$, define the operator U by $U(f + g) = U_1 f + U_2 g$ for $f \in \mathcal{X}_\infty$ and $g \in \mathcal{Y}_\infty$. By the construction, U is a well defined unitary operator on $\mathcal{X}_\infty \oplus \mathcal{Y}_\infty$. Clearly $z(n) = U^n z(0)$ is a weakly stationary process.

By definition $\hat{x}(n) \in \mathcal{X}_{n-1}$, so there exist $\{a_k^{(j)} : -N_j \le k \le n-1,\ j \ge 1\}$ such that $\hat{x}(n) = \lim_{j\to\infty} \sum_{k=-N_j}^{n-1} a_k^{(j)} x(k)$, where the limit is in the mean. Since x and y have the same spectral measure,

$$\left\| \sum_{k=-N_j}^{n-1} a_k^{(j)} x(k) - \sum_{k=-N_m}^{n-1} a_k^{(m)} x(k) \right\|^2 = \left\| \sum_{k=-N_j}^{n-1} a_k^{(j)} y(k) - \sum_{k=-N_m}^{n-1} a_k^{(m)} y(k) \right\|^2$$

for every j and m, which yields that the (sequence of) vectors $\sum_{k=-N_j}^{n-1} a_k^{(j)} y(k)$ converges as $j \to \infty$ to a limit $w(n) \in \mathcal{Y}_{n-1}$. Since x and y have the same spectral measure and the inner product is continuous, for every $l \le n-1$ we have

$$0 = \langle x(n) - \hat{x}(n), x(l) \rangle = \lim_{j\to\infty} \langle x(n) - \sum_{k=-N_j}^{n-1} a_k^{(j)} x(k), x(l) \rangle$$

$$= \lim_{j\to\infty} \langle y(n) - \sum_{k=-N_j}^{n-1} a_k^{(j)} y(k), y(l) \rangle = \langle y(n) - w(n), y(l) \rangle.$$

The uniqueness of the orthogonal projection yields $\hat{y}(n) = w(n)$. By the representation of $\hat{y}(n)$ we have

$$\hat{x}(n) + \hat{y}(n) = \lim_{j\to\infty} \sum_{k=-N_j}^{n-1} a_k^{(j)} [x(k) + y(k)] = \lim_{j\to\infty} \sum_{k=-N_j}^{n-1} a_k^{(j)} z(k);$$

hence $\hat{x}(n) + \hat{y}(n) \in \mathcal{Z}_{n-1}$ (note that in general $\mathcal{Z}_{n-1} \subset \mathcal{X}_{n-1} \oplus \mathcal{Y}_{n-1}$ with strict inclusion). By the orthogonality of x and y, for every $l \le n-1$ we have

$$\langle z(n) - (\hat{x}(n) + \hat{y}(n)), z(l) \rangle = \langle (x(n) - \hat{x}(n)) + (y(n) - \hat{y}(n)), x(l) + y(l) \rangle = 0.$$

The uniqueness of the orthogonal projection yields that $\hat{z}(n) = \hat{x}(n) + \hat{y}(n)$.

(ii)–(iii) As noted in the discussion following Lemma 6.5, $\sigma = \|x(n) - \hat{x}(n)\|$ for any n. By the representation of $\hat{y}(n)$ and the assumption that x and y have the same spectral measure, also $\|y(n) - \hat{y}(n)\| = \sigma$ for every n, so $\|z(n) - \hat{z}(n)\| = \sigma\sqrt{2}$. Define $\mathcal{I}_\infty^x := c.l.m.\{\xi(n) : n \in \mathbb{Z}\}$, $\mathcal{I}_\infty^y = c.l.m.\{\eta(n) : n \in \mathbb{Z}\}$, and $\mathcal{I}_\infty^z = c.l.m.\{z(n) - \hat{z}(n) : n \in \mathbb{Z}\}$. The purely-indeterministic components, x_u, y_u, z_u, are the orthogonal projections of x, y, z, on $\mathcal{I}_\infty^x$, $\mathcal{I}_\infty^y$, $\mathcal{I}_\infty^z$, respectively (see [5, p. 572]). The orthogonality of x and y and the equality $\hat{z}(n) = \hat{x}(n) + \hat{y}(n)$ yield that for any k,

$$\langle z(n), z(k) - \hat{z}(k) \rangle = \langle x(n), x(k) - \hat{x}(k) \rangle + \langle y(n), y(k) - \hat{y}(k) \rangle,$$

with all inner products zero for $n < k$. Since x and y have the same spectral measure we have $\langle x(n), x(k) - \hat{x}(k) \rangle = \langle y(n), y(k) - \hat{y}(k) \rangle$. By writing explicitly the orthogonal projection of $z(n)$ on $\mathcal{I}_\infty^z$, i.e., the Fourier series of $z(n)$ relative to the orthogonal system $\{z(n) - \hat{z}(n)\}$, we obtain (ii) and (iii). □

Let $\{x_u^{(1)}(n)\}$ ($\{x_u^{(2)}(n)\}$) and $\{x_v^{(1)}(n)\}$ ($\{x_v^{(2)}(n)\}$) be the purely-indeterministic and the deterministic parts in the Wold decomposition (over $\mathbb{R}$) of the real stationary process $\{x^{(1)}(n)\}$ ($\{x^{(2)}(n)\}$), respectively. By the construction of the Wold decomposition, these real processes are stationary, and belong to the closed subspace (over the reals) generated by $\{x^{(1)}(n)\}$ ($\{x^{(2)}(n)\}$).

Corollary 6.3. *With the above notations, the Wold decomposition (over $\mathbb{C}$) of $x = x^{(1)} - ix^{(2)}$ is $x_u = x_u^{(1)} - ix_u^{(2)}$ and $x_v = x_v^{(1)} - ix_v^{(2)}$.*

Proof. Apply Lemma 6.2 (over the complex field), and use Lemma 6.1 for the Wold decompositions of $x^{(1)}$ and $x^{(2)}$. □

The Wold decomposition of a homogeneous *real* random field can be defined in analogy to [16], including the notion of an evanescent homogeneous real random field (with respect to a given order). The geometric procedure to avoid the use of the spectral theorem is similar to the above. In particular, a deterministic real random field with respect to the lexicographic order is defined in the real case as in the introduction (but over the real field); for any RNSHP order, a deterministic real random field is similarly defined.

Proposition 6.4. *The real random field $\tilde{e}(n, m)$ is deterministic with respect to any RNSHP.*

Proof. First we consider the order induced by a and b. If $\tilde{e}(n, m)$ is predictable with respect to the past, the points that contribute to the prediction are on the ray that comes out from (n, m) with a slope of $-a/b$ (we can not expect the samples $\{x^{(1)}(k),\ x^{(2)}(k) : k < na + mb\}$ to contribute to the prediction, for example if $x^{(1)}$ and/or $x^{(2)}$ are purely-indeterministic.)

Precisely, let $\epsilon = \operatorname{sign}(a)$; clearly $(n + \epsilon jb, m - \epsilon ja) \prec (n, m)$ for $j = 1, 2, \ldots$. We would like to show the existence of *real* coefficients $\{a_j^{(k)}\}_{j=1}^{N_k}$ such that $\tilde{e}(n, m) = \lim_{k\to\infty} \sum_{j=1}^{N_k} a_j^{(k)} \tilde{e}(n+\epsilon jb, m-\epsilon ja)$, where the limit is in the mean.

The (square of the) estimation error at the point (n, m) is defined by

$$d_{n,m}^{(k)} := \|\tilde{e}(n, m) - \sum_{j=1}^{N_k} a_j^{(k)} \tilde{e}(n + \epsilon jb, m - \epsilon ja)\|^2.$$

Putting $\rho = \operatorname{Arg}(\lambda)$ and using orthogonality of $x^{(1)}$ and $x^{(2)}$, we obtain

$$\begin{aligned} d_{n,m}^{(k)} &= \|x^{(1)}(na + mb)\|^2 |\cos((nc + md)\rho) - \sum_{j=1}^{N_k} a_j^{(k)} \cos((nc + md - \epsilon j)\rho)|^2 \\ &\quad + \|x^{(2)}(na + mb)\|^2 |\sin((nc + md)\rho) - \sum_{j=1}^{N_k} a_j^{(k)} \sin((nc + md - \epsilon j)\rho)|^2. \end{aligned}$$

Let $\sigma = \|x^{(1)}(0)\| = \|x^{(2)}(0)\|$; then

$$d_{n,m}^{(k)} = \sigma^2 |\mathrm{e}^{i(nc+md)\rho} - \sum_{j=1}^{N_k} a_j^{(k)} \mathrm{e}^{i(nc+md-\epsilon j)\rho}|^2 = \sigma^2 |1 - \sum_{j=1}^{N_k} a_j^{(k)} \mathrm{e}^{-i\epsilon j\rho}|^2.$$

Now we distinguish two cases: (i) $j_0\rho = 2\pi l$ for some integers j_0 and l; (ii) $2\pi/\rho$ is irrational.

(i): We define $a_{j_0}^{(k)} = 1$ and otherwise $a_j^{(k)} = 0$, for any $k \geq 1$.

(ii): For each k we define $N_k = k$; for $1 \leq j \leq k$ define $a_j^{(k)} = \frac{2\cos(\epsilon j\rho)}{k}$, otherwise $a_j^{(k)} = 0$. We have

$$d_{n,m}^{(k)} = \sigma^2 \Big| 1 - \sum_{j=1}^{k} \frac{\mathrm{e}^{-i\epsilon j\rho} + \mathrm{e}^{i\epsilon j\rho}}{k} \cdot \mathrm{e}^{-i\epsilon j\rho} \Big|$$

$$\sigma^2 = \Big| \frac{\mathrm{e}^{-2i\epsilon\rho}}{k} \cdot \frac{1-\mathrm{e}^{-2i\epsilon\rho k}}{1-\mathrm{e}^{-2i\epsilon\rho}} \Big| \leq \frac{2\sigma^2}{k|1-\mathrm{e}^{-2i\epsilon\rho}|} \underset{k\to\infty}{\to} 0. \tag{18}$$

Let α and β be the parameters of another RNSHP order, with past $P_{\alpha,\beta}$ defined by (2). Since the intersection $P_{\alpha,\beta} \cap P_{a,b}$ contains a ray (half line) of the boundary line of $P_{a,b}$, it is easy to show that the method of approximation which was shown above holds, and the field $\tilde{e}$ is deterministic with respect to any RNSHP. □

We now define the following closed subspaces (over the reals):

$$\begin{aligned}
\mathcal{E}_{n,m} &= c.l.m.\{\tilde{e}(p,q) : (p,q) \preceq (n,m)\},\\
\mathcal{X}_n &= c.l.m.\{x^{(1)}(k), x^{(2)}(k) : k \leq n\},\\
\mathcal{E}_\infty &= c.l.m.\{\tilde{e}(n,m) : (n,m) \in \mathbb{Z}^2\},\\
\mathcal{X}_\infty &= c.l.m.\{x^{(1)}(n), x^{(2)}(n) : n \in \mathbb{Z}\},\\
\mathcal{E}_{-\infty} &= \bigcap_{n,m\in\mathbb{Z}} \mathcal{E}_{n,m}, \quad \mathcal{X}_{-\infty} = \bigcap_{n\in\mathbb{Z}} \mathcal{X}_n.
\end{aligned}$$

Similarly, we define $\mathcal{X}_n^{(1)}$, $\mathcal{X}_\infty^{(1)}$ and $\mathcal{X}_{-\infty}^{(1)}$ for $x^{(1)}$, and $\mathcal{X}_n^{(2)}$, $\mathcal{X}_\infty^{(2)}$ and $\mathcal{X}_{-\infty}^{(2)}$ for $x^{(2)}$. The orthogonality of $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$ means $\mathcal{X}_\infty^{(1)} \perp \mathcal{X}_\infty^{(2)}$.

Lemma 6.5. *With the above notations, we have*

$$\mathcal{E}_\infty = \mathcal{X}_\infty^{(1)} \oplus \mathcal{X}_\infty^{(2)} \quad \text{and} \quad \mathcal{E}_{-\infty} = \mathcal{X}_{-\infty}^{(1)} \oplus \mathcal{X}_{-\infty}^{(2)}$$

Proof. Since $(p,q) \preceq (n,m)$ implies $pa+qb \leq na+mb$ we have $\mathcal{E}_{n,m} \subset \mathcal{X}_{na+mb}$, and by orthogonality, $\mathcal{X}_{na+mb} = \mathcal{X}_{na+mb}^{(1)} \oplus \mathcal{X}_{na+mb}^{(2)}$. Hence $\mathcal{E}_{n,m} \subset \mathcal{X}_{na+mb}^{(1)} \oplus \mathcal{X}_{na+mb}^{(2)}$.

To obtain the converse inclusion, we start by noting that for $2\pi/\rho$ irrational and any integers $m_1 \neq 0$ and m_2, we have

$$\frac{1}{k} \sum_{j=1}^{k} \cos[m_1(m_2+j)\rho] \underset{k\to\infty}{\to} 0. \tag{19}$$

The proof is along the same lines of the computation of (18); the same is true for the averages with sine functions.

We distinguish two cases: (i) $j_0\rho = 2\pi l$ for some integers j_0 and l; (ii) $2\pi/\rho$ is irrational.

(i): Let $(p,q) \preceq (n,m)$, surely there exists $j_1 \geq 0$ such that $(pc+qd-\epsilon j_1) = \pm l_1 j_0$ for some natural $l_1 \neq 0$. The point $(p+\epsilon j_1 b, q-\epsilon j_1 a) \preceq (p,q) \preceq (n,m)$ therefore, $x^{(1)}(pa+qb) = \tilde{e}(p+\epsilon j_1 b, q-\epsilon j_1 a) \in \mathcal{E}_{n,m}$, and from that also $x^{(2)}(pa+qb) \in \mathcal{E}_{n,m}$. Since $(p,q) \preceq (n,m)$ is an arbitrary point, $\mathcal{X}^{(1)}_{na+mb}, \mathcal{X}^{(2)}_{na+mb} \subset \mathcal{E}_{n,m}$, and therefore $\mathcal{X}^{(1)}_{na+mb} \oplus \mathcal{X}^{(2)}_{na+mb} \subset \mathcal{E}_{n,m}$.

(ii): The idea of the proof is to "demodulate $\{\tilde{e}(n,m)\}$ by its harmonic carrier". For $(p,q) \preceq (n,m)$ we compute

$$\begin{aligned}
S^{(k)}_{p,q} :&= \sum_{j=1}^{k} \tilde{e}(p+\epsilon jb, q-\epsilon ja)\cos[(pc+qd-\epsilon j)\rho] \\
&= x^{(1)}(pa+qb)\sum_{j=1}^{k}\cos^2[(pc+qd-\epsilon j)\rho] \\
&\quad + x^{(2)}(pa+qb)\sum_{j=1}^{k}\sin[(pc+qd-\epsilon j)\rho]\cos[(pc+qd-\epsilon j)\rho] \\
&= \frac{1}{2}x^{(1)}(pa+qb)\Big(k+\sum_{j=1}^{k}\cos[2(pc+qd-\epsilon j)\rho]\Big) \\
&\quad +\frac{1}{2}x^{(2)}(pa+qb)\sum_{j=1}^{k}\sin[2(pc+qd-\epsilon j)\rho].
\end{aligned}$$

By (19) (and the same for series of sines) $\lim_{k\to\infty}\frac{S^{(k)}_{p,q}}{k} = \frac{x^{(1)}(na+mb)}{2}$ (everywhere), therefore $x^{(1)}(pa+qb) \in \mathcal{E}_{n,m}$; the same arguments as in (i) yield that $\mathcal{X}^{(1)}_{na+mb} \oplus \mathcal{X}^{(2)}_{na+mb} \subset \mathcal{E}_{n,m}$.

Using the fact, already proved above, that $\mathcal{E}_{n,m} = \mathcal{X}^{(1)}_{na+mb} \oplus \mathcal{X}^{(2)}_{na+mb}$, we conclude that $\mathcal{E}_\infty = \mathcal{X}_\infty = \mathcal{X}^{(1)}_\infty \oplus \mathcal{X}^{(2)}_\infty$. Since

$$\begin{aligned}
\cap_{n,m}\mathcal{E}_{n,m} &= \cap_{n,m}(\mathcal{X}^{(1)}_{na+mb} \oplus \mathcal{X}^{(2)}_{na+mb}) \\
&= (\cap_{n,m}\mathcal{X}^{(1)}_{na+mb}) \oplus (\cap_{n,m}\mathcal{X}^{(2)}_{na+mb}) = (\cap_s \mathcal{X}^{(1)}_s) \oplus (\cap_s \mathcal{X}^{(2)}_s)
\end{aligned}$$

we obtain $\mathcal{E}_{-\infty} = \mathcal{X}_{-\infty} = \mathcal{X}^{(1)}_{-\infty} \oplus \mathcal{X}^{(2)}_{-\infty}$. □

Proposition 6.6. *Let $x^{(1)}$ and $x^{(2)}$ be real weakly stationary processes and put $x = x^{(1)} - ix^{(2)}$. Then the following are equivalent:*

(i) The field $e(n,m) = x(na+mb)\lambda^{nc+md}$ is evanescent with respect to the order induced by a and b.
(ii) $x^{(1)}$ and $x^{(2)}$ are both purely-indeterministic.
(iii) $x^{(1)}$ or $x^{(2)}$ is purely-indeterministic.
(iv) The real random field $\tilde{e}(n,m) = \Re\{x(na+mb)\lambda^{nc+md}\}$ is evanescent with respect to the order induced by a and b.

Proof. $x^{(1)}$ is purely-indeterministic if and only if its spectral measure is absolutely continuous. Since $x^{(1)}$ and $x^{(2)}$ have the same spectral measure, $x^{(1)}$ is purely-indeterministic if and only if $x^{(2)}$ is. This shows (ii) $\Leftrightarrow$ (iii).

By Proposition 6.4, $\tilde{e}$ is deterministic. By definition, it is evanescent if and only if $\tilde{e} \perp \mathcal{E}_{-\infty}$. By the representation (17) of $\tilde{e}$ and the equality $\mathcal{E}_{-\infty} = X^{(1)}_{-\infty} \oplus \mathcal{X}^{(2)}_{-\infty}$, proved in Lemma 6.5, $\tilde{e}$ is orthogonal to $\mathcal{E}_{-\infty}$ if and only if $\{x^{(1)}(n)\}$ is orthogonal to $\mathcal{X}^{(1)}_{-\infty}$ and $\{x^{(2)}(n)\}$ is orthogonal to $\mathcal{X}^{(2)}_{-\infty}$, that is, if and only if $x^{(1)}$ and $x^{(2)}$ are both purely-indeterministic. By Corollary 6.3 the last statement is true if and only if x is purely-indeterministic, and by Proposition 2.1 x is purely-indeterministic if and only if e is evanescent. □

Remarks. In analogy to Corollary 2.2, one can obtain a similar corollary for the real case.

Theorem 6.7. *The real random field $\ddot{e}$ determined, in analogy to (17), by $\{x^{(1)}_u(n)\}$ and $\{x^{(2)}_u(n)\}$ (the purely-indeterministic parts of the one-dimensional Wold decompositions of $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$), is evanescent, and is precisely the evanescent part of $\{\tilde{e}(n, m)\}$, (all with respect to the order induced by a and b).*

Proof. Since $x^{(1)}$ and $x^{(2)}$ are assumed orthogonal, also $\{x^{(1)}_u, x^{(1)}_v\} \perp \{x^{(2)}_u, x^{(2)}_v\}$. Since $x^{(1)}$ and $x^{(2)}$ have the same spectral measure, by the uniqueness of Lebesgue's decomposition so do $x^{(1)}_u$ and $x^{(2)}_u$ ($x^{(1)}_v$ and $x^{(2)}_v$); hence the real fields determined, in analogy to (17), by $(x^{(1)}_u, x^{(2)}_u)$ or by $(x^{(1)}_v, x^{(2)}_v)$, are well defined homogeneous random fields, and $\ddot{e}$ is evanescent by Proposition 6.6.

Lemma 6.5 asserts that $\mathcal{E}_{\infty} = \mathcal{X}^{(1)}_{\infty} \oplus \mathcal{X}^{(2)}_{\infty}$, so all the processes are in $\mathcal{E}_{\infty}$. By the Wold decomposition,

$$x^{(1)}_v(na+mb)\cos[(nc+md)\rho]+x^{(2)}_v(na+mb)\sin[(nc+md)\rho] \in \mathcal{X}^{(1)}_{-\infty} \oplus \mathcal{X}^{(2)}_{-\infty}.$$

By the Wold decomposition $x^{(1)}_u(n) \perp \mathcal{X}^{(1)}_{-\infty}$ and $x^{(2)}_u(n) \perp \mathcal{X}^{(2)}_{-\infty}$; using the orthogonality assumptions $x^{(1)}_u(n) \perp \mathcal{X}^{(2)}_{-\infty}$ and $x^{(2)}_u(n) \perp \mathcal{X}^{(1)}_{-\infty}$; hence (in $\mathcal{E}_{\infty}$) we have

$$x^{(1)}_u(na+mb)\cos[(nc+md)\rho]+x^{(2)}_u(na+mb)\sin[(nc+md)\rho] \in (\mathcal{X}^{(1)}_{-\infty}\oplus\mathcal{X}^{(2)}_{-\infty})^{\perp}.$$

Let $\tilde{e} = \tilde{e}_1 + \tilde{e}_2$ be the two-dimensional Wold decomposition of $\tilde{e}$, where the orthogonal $\tilde{e}_1$ and $\tilde{e}_2$ are the evanescent and the remote past parts, respectively.

Since $\{\tilde{e}_2(n, m)\} \subset \mathcal{E}_{-\infty}$ and $\mathcal{E}_{-\infty} = \mathcal{X}^{(1)}_{-\infty} \oplus \mathcal{X}^{(2)}_{-\infty}$, the uniqueness of the direct sum gives the result. □

Remarks. By Corollary 6.3 $x_u = x^{(1)}_u - i x^{(2)}_u$, and by Theorem 2.3, we have that $x_u(na+mb)\lambda^{nc+md}$ is a complex evanescent field. Using Theorem 6.7 we conclude that for the field defined by $e(n, m) = [x^{(1)}(na + mb) - i x^{(2)}(na + mb)]\lambda^{nc+md}$, the following statement is true: *the real part of the evanescent component of the field is the evanescent component of the real part of the field.*

Note that in general, the real part of a homogenous complex random field is not necessarily homogenous, so that the above statement is not necessarily true for general homogenous fields of the form (1).

Spectral analysis. Let $\mathcal{H}^{(1)} := \mathcal{X}_\infty^{(1)} + i\mathcal{X}_\infty^{(1)}$ and $\mathcal{H}^{(2)} := \mathcal{X}_\infty^{(2)} + i\mathcal{X}_\infty^{(2)}$ be the closed linear manifolds over the *complex* field, generated by $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$ respectively, and let U_1 and U_2 be the corresponding unitary operators, defined on $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ respectively. The orthogonality of $x^{(1)}$ and $x^{(2)}$ means $\mathcal{H}^{(1)} \perp \mathcal{H}^{(2)}$, so we can define on the direct sum $\mathcal{H} := \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$ a unitary operator $U = U_1 \oplus U_2$. Let E be the resolution of the identity for U; clearly, by construction we have $E = E_1 \oplus E_2$ where E_1 and E_2 are the corresponding resolutions of the identity of U_1 and U_2 respectively. Let $w_1(\cdot) = E(\cdot)x^{(1)}(0),\ w_2(\cdot) = E(\cdot)x^{(2)}(0)$ be the stochastic (i.e., $\mathcal{H}$-valued) measures corresponding to $\{x^{(1)}(n)\}$ and $\{x^{(2)}(n)\}$ respectively; note that these two measures are orthogonal. Denote by $\mu = \|w_1\|_{\mathcal{H}}^2 = \|w_2\|_{\mathcal{H}}^2$ the common spectral measure of $x^{(1)}$ and $x^{(2)}$. By writing

$$\tilde{e}(n,m) = \frac{1}{2}\{x^{(1)}(na+mb)(\lambda^{nc+md} + \overline{\lambda}^{nc+md}) \\ -ix^{(2)}(na+mb)(\lambda^{nc+md} - \overline{\lambda}^{nc+md})\},$$

the method that led to Theorem 3.2 now leads to

Theorem 6.8. *The stochastic measure corresponding to $\tilde{e}$ has the form*

$$\frac{1}{2}\{w_1 \times (\delta_\lambda + \delta_{\overline{\lambda}}) - iw_2 \times (\delta_\lambda - \delta_{\overline{\lambda}})\} \circ T^{-1}$$

Now, using Theorem 6.8 we obtain the spectral measure of $\tilde{e}$. Using the facts that w_1 and w_2 are orthogonal, with the same spectral measure, and that $(\delta_\lambda + \delta_{\overline{\lambda}})^2 + (\delta_\lambda - \delta_{\overline{\lambda}})^2 = 2(\delta_\lambda + \delta_{\overline{\lambda}})$, we have

Theorem 6.9. *The spectral measure of $\tilde{e}$ has the form*

$$\frac{1}{2}\mu \times (\delta_\lambda + \delta_{\overline{\lambda}}) \circ T^{-1}$$

The above theorem has an analogous formulation in the $[0, 2\pi]^2$ space. Its geometric interpretation is that the spectral measure of $\tilde{e}$ is concentrated on two parallel straight lines satisfying the equations $\nu a - \omega b = \pm\rho$. Using similar techniques, all the other results derived in the previous sections for complex valued fields can be adapted to the case of real valued fields; for example in analogy to Theorem 4.2.

Theorem 6.10. *The spectral pseudo-density σ of the real process $\tilde{e}$ has the form*

$$\sigma(\omega,\nu) = \tilde{S}(\omega,\nu) \cdot \delta(\{\rho - (\nu a - \omega b)\}) \\ + \tilde{\tilde{S}}(\omega,\nu) \cdot \delta(\{\rho + (\nu a - \omega b)\})$$

where δ is the Dirac measure concentrated at the origin, and $\tilde{S}, \tilde{\tilde{S}}$ are continuous periodic functions on $[0, 2\pi]^2$, such that their restrictions to the lines $\rho = \nu a - \omega b$ and $-\rho = \nu a - \omega b$, respectively, are symmetric with respect to $(\omega, \nu) = (0, 0)$. Furthermore, the function σ is uniquely determined, independently of $\{(n_k, m_k)\}$.

Remarks. As a consequence of Theorem 4.3 (see Remark 1 following it), for a real valued process the *middle point* ($h = 0$) is also a symmetry point of the spectral pseudo-density in the one-dimensional coordinate system.

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