PARAMETRIC ESTIMATION OF TWO-DIMENSIONAL AFFINE TRANSFORMATIONS

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ABSTRACT

We consider the general problem of object recognition based on a set of known templates. While the set of templates is known, the tremendous set of possible transformations and deformations between the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. We propose a method that reduces the high dimensional problem of evaluating the orbit created by applying the set of all possible transformations in the group to a template, into a problem of analyzing a function in a low dimensional Euclidian space. In this setting, the problem of estimating the parametric model of the affine deformation is expressed using a set on non-linear operators, by a set of *linear* equations. This system of linear equations is then solved for the transformation parameters.

1. INTRODUCTION

This paper is concerned with the general problem of object recognition based on a set of known templates. However, while the set of templates is known, the variability associated with the object, such as its location and pose in the observed scene, or its deformation are unknown *a-priori*, and only the group of actions causing this variablity in the observation, can be defined. This huge variability in the object signature (for any single object) due to the tremendous set of possible transformations and deformations between the template and the observed signature, makes any detection and recognition problem ill-defined unless this variablity is taken into account. In other words, estimation of the transformation of the object with respect to any template in an indexed set is an inherent and essential part of any detection and recognition problem.

The fundamental settings of the problem known as *deformable templates* are set in [1] and the references therein. There are two key elements in a deformable template representation: A typical element (the template); and a family of transformations and deformations which when applied to the typical element produces other elements. In the simplest case where rigid objects are observed the transformations are composed of translation, scaling and rotation.

This work was supported by the EU 5th Framework IHP Program, MOUMIR Project, under Grant RTN-1999-0177.

The action of the transformation/deformation on the set of known templates forms a group action on the space of all possible transformed templates. Thus each template is represented by its orbit, induced by the action of the group on the template. For multiple objects, the transformation space is best described as a union of orbits, each representing a different object and its possible deformations. Thus, given measurements of an observed object (for example, in the form of an image) recognition becomes the procedure of jointly finding the group element and object template that minimize some metric with respect to the observation.

To enable a rigorous treatment of the problem we begin by defining the "similarity criterion". Let G be a group and S be a set (a function space in our case), such that G acts as a transformation group on S. We define a geometry on the function space S, and an affine group G, where the action of G on S is defined by $G \times S \rightarrow S$ such that for every $\phi \in G$ and every $s \in S$, $(\phi, s) \rightarrow s \circ \phi$ (composition of functions on the right), where $s \circ \phi \in S$. From this point of view, given two functions h and g on the same orbit, the initial task (that enables recognition in a second stage), is to find the element ϕ in G that makes h and g similar in the sense that $h = g \circ \phi$. To simplify the discussion, at this stage we exclude the case of self-similar functions, *i.e.*, functions $f \in S$ for which there exists some $\phi \in G$ such that ϕ is not the identity element while $f = f \circ \phi$.

To better illustrate the approach consider the following examples:

1. $S_1 = \{f : R \rightarrow R | f \text{ measureable and bounded}\}$, $G_1 = \{\varphi : R \rightarrow R | \varphi(x) = ax + b \ , \ a \neq 0\}$.
   In this setting the objects are measurable and bounded functions from the real line to itself and the group is the group of linear non-singular transformations. (Non-singularity is essential for having the structure of a group). Let s be an object in S_1. The group action defined by G_1 implies that the only transformations s can undergo are uniform scaling of its x-axis and shift. Consider now the entire family $\{s \circ \varphi | \varphi \in G_1\}$ which is the family of functions induced by the single object s (the orbit). Thus, given some other element in this family we would like to know the parameters of its transformation relative to s. If a time dependent sequence is provided, we would like to track the object evolvement as a function of time.
2. $S_3 = \left\{f : R^2 \rightarrow R | f \text{ measureable and bounded}\right\}$, $G_3 = \left\{\varphi : R^2 \rightarrow R^2 | \varphi \in SO_2(R)\right\}$.

This is one of the simplest examples when two-dimensional domains are being considered. In this example the group action model only allows the objects to be rotated or translated from their "original" position. Thus the objects in this framework are all "rigid".

The above setting enables us to formalize practical questions using a rigorous setting:

1. Motion Analysis: Is the problem of estimating the group action as a function of time so that at each instant the correct group element that acts on the object is identified. (In some cases it is possible to add topological structure to the group or even an analytic structure (Lie groups) that enables the tracking of continuous or differentiable motions.)
2. Detection and Recognition: Is the task of finding the equivalence class of the given object, thus identifying the orbit associated with this object.

In this paper we concentrate on the analysis of parametric modeling and estimation of affine transformations. (This problem is a special case of the general problem of modeling the homeomorphisms group). Theoretically, in the absence of noise, the solution to the recognition problem is obtained by applying each of the deformations in the group to the template, followed by comparing the result to the observed realization. In the absence of noise, one of the deformations is identical to the observation. Thus the procedure of searching for the deformation that transforms g into h is achieved in principle by a mapping from the group (the affine group, in our case) to the space of functions defined by the orbit of g. However, as the number of such possible deformations is infinite, this direct approach is computationally prohibitive. Hence, more sophisticated methods are essential.

The center of the proposed solution is a method to reduce the high dimensional problem of evaluating the orbit created by applying the set of transformations in $GL(n)$ into a problem of analyzing a function in the vector space R^n. In this paper we prove that the proposed approach leads to an exact, closed form solution for the problem of estimating the affine transformation.

2. THE AFFINE TRANSFORMATION

We begin by defining the geometry of the transformation for the case where the transformed objects are two-dimensional. Extensions to the n-dimensional case are immediate, as we show throughout. More specifically, let D be a compact subset of $R \times R$ and let $f : D \to R$ be an integrable function. Let also $\mathrm{GL}_2(R)$ denote the group of real valued invertible 2×2 matrices. Let $\mathbf{A}$ be some matrix in $\mathrm{GL}_2(R)$ and let $\mathbf{s}$ be a two-dimensional vector representing the shift operation. Applying the transformation $\mathbf{A}$ to every $(x, y) \in D$ followed by shifting the result by $\mathbf{s}$, defines a new compact subset of $R\times R$, that we denote by D_A. Figure 1 illustrates the result of applying such a transformation, with $\mathbf{s} = \mathbf{0}$, to the basis vectors $[1, 0]^T$, $[0, 1]^T$, and the result of applying the affine transformation $\mathbf{A}$ to D, where the transformed basis vectors are now given by $[a_{11}, a_{21}]^T$, $[a_{12}, a_{22}]^T$, respectively.

In the following we will be interested in obtaining the representation of the "surface shape" $f(x, y), (x, y) \in D$

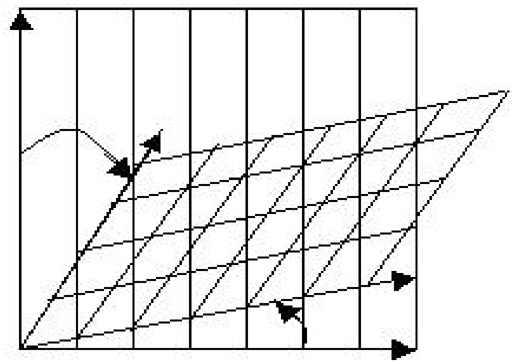

Figure 1: Two-dimensional affine transformation.

in the orthogonal coordinate system defined on D_A. In the orthogonal coordinate system defined on D_A the *same* surface is described by the function $g : D_A \to R$. Thus, $g(x', y') = g(a_{11}x+a_{12}y+s_1, a_{21}x+a_{22}y+s_2) = f(x, y)$, or more generally, $g(x', y') = f(\mathbf{A}^{-1}[\mathbf{x}' - \mathbf{s}'])$. To simplify the notations we shall assume without limiting the generality of the derivation that $\mathbf{s}' = \mathbf{0}$. In this paper we consider the problem of estimating $\mathbf{A}$ given the observations on $g(x', y')$ and $f(x, y)$.

3. PROBLEM DEFINITION

The general problem addressed in this paper is the following: Given two bounded measurable functions h, g with compact support, and with no affine symmetry (as rigorously defined below), such that

$$h : R^n \to R$$
$$g : R^n \to R$$

where

$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) \ , \quad \mathbf{A} \in GL_n(R), \;\; \mathbf{x} \in R^n \qquad (1)$$

find the matrix $\mathbf{A}$. We note that both h and g are defined on spaces of the same dimension, *i.e.*, no projection into a lower dimensional space is involved.

Let $M(R^n, R)$ denote the space of measurable functions from R^n to R, and let $\mathbf{x}$ denote a vector in R^n. Let $N \subset M$ denote the set of measurable functions with an affine symmetry (or affine invariance), *i.e.*, $N = \{f \in M(R^n, R) | \exists \mathbf{A} \in \mathrm{GL}_n(R), \mathbf{A} \neq \mathbf{I}, f(\mathbf{x}) = f(\mathbf{A}\mathbf{x})\}$ for every $\mathbf{x} \in R^n$. (Thus, N is the "stabilizer" of $M(R^n, R)$ under $\mathrm{GL}_n(R)$. Obviously for any $f \in N$ and $\mathbf{A}$ such that $f(\mathbf{x}) = f(\mathbf{A}\mathbf{x})$ for every $\mathbf{x} \in R^n$, $\mathbf{A}$ cannot be uniquely recovered).

To illustrate the notion of N consider the following example: Let

$$f(x) = \begin{cases} 1 & x \in [2^n, 2^n + 2^{n-1}] \;\; \forall n \in Z \\ 0 & else \end{cases}$$

This function satisfies the relation $f(x) = f(2x)$ but clearly there is no hope to make any distinction between $f(A_1 x) = f(2A_1 x)\ \forall A_1 \in R$. Hence, $f \in N$.

Other examples of affine invariant functions include any constant function defined on all of R^n; any periodic function defined on all of R^n; and in the two dimensional case, the functions with radial symmetry (as $\mathrm{SO}_2(R) \subset \mathrm{GL}_2(R)$).

Let $M_{Aff}(R^n, R) \triangleq M(R^n, R) \setminus N$ denote the set of measurable functions with no affine symmetry. Next, partition $M_{Aff}(R^n, R)$ into affine equivalence sets by the equivalence relation $f \sim g \Leftrightarrow \exists \mathbf{A} \in GL_n(R) \,|\, f(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$. (It can be easily checked that this is indeed an equivalence relation). Denote the quotient space by $Q_{Aff}(R^n, R)$, and let $[f]$ denote the equivalence set of f.

Lemma 1 *Let $g \in M_{Aff}(R^n, R)$. Assume $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$. Then the transformation $\mathbf{A} \in GL_n(R)$ satisfying the relation $f(x) = g(\mathbf{A}\mathbf{x})$, is unique.*

We can now rigorously formalize the scope of the problem addressed in this paper as follows: We provide *an exact, closed form solution* to the problem of estimating the affine transformation $\mathbf{A} \in GL_n(R)$ for any two objects f and g that satisfy an affine relation, *i.e.*, $[f] = [g] = q$, for any $q \in Q_{Aff}(R^n, R)$. An equivalent statement of the estimation problem is as follows: Given two functions h and g such that

$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) \qquad \mathbf{A} \in GL_n(R) \tag{2}$$

find $\mathbf{A}$.

Finally, we note that in the following derivation it is assumed that the functions are bounded and have compact support (as they are measurable but not necessarily continuous). It is further assumed that $\mathbf{A} \in GL_n(R)$ has a positive determinant.

4. AN ALGORITHMIC SOLUTION

In this section we provide a constructive proof showing that given an observation on $h(\mathbf{x})$ and an observation on $g(\mathbf{x})$, $\mathbf{A}$ can be *uniquely* estimated. It is further shown how this estimator is implemented.

Let $\mathbf{x}, \mathbf{y} \in R^n$, *i.e.*,

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$$

$$\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$$

where

$$\mathbf{y} = \mathbf{A}\mathbf{x}\ ,\ \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

and

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}$$

Since $\mathbf{A} \in GL_n(R)$, also $\mathbf{A}^{-1} \in GL_n(R)$. It is therefore possible to solve for $\mathbf{A}^{-1}$ and the solution for $\mathbf{A}$ is guaranteed to be in $GL_n(R)$. Moreover, as shown below, in the proposed procedure the transformation determinant is evaluated first, and by a different procedure than the one employed to estimate the elements of $\mathbf{A}^{-1}$. Hence, a nonzero estimate of the Jacobian guarantees the existence of an inverse of the transformation matrix. More specifically, let

$$h, g, f : R^n \to R$$

and define the notation

$$\int_{R^n} f = \int_R \cdots \int_R f dx_1 dx_2 \cdots dx_n$$

The first step in the solution is to find the determinant of the matrix $\mathbf{A}$. A simple approach is to evaluate the Jacobian through the identity relation:

$$\int_{R^n} h^2(\mathbf{x}) = \int_{R^n} g^2(\mathbf{A}\mathbf{x}) = \left|\mathbf{A}^{-1}\right| \int_{R^n} g^2(\mathbf{x}) \tag{3}$$

or through similar identities. (The specific choice made in (3) is motivated by the convenience of handling functions in L_2.) Hence,

$$\left|\mathbf{A}^{-1}\right| = \frac{\int_{R^n} h^2(\mathbf{x})}{\int_{R^n} g^2(\mathbf{x})} \tag{4}$$

and $\left|\mathbf{A}^{-1}\right| = |\mathbf{A}|^{-1}$. In the second stage, n linear and independent constrains on the matrix elements must be set. More specifically, let $(\mathbf{A}^{-1})_k$ denote the kth row of $(\mathbf{A}^{-1})$. We then have

$$\begin{aligned} \int_{R^n} x_k h(\mathbf{x}) &= \int_{R^n} x_k g(\mathbf{A}\mathbf{x}) = \left|\mathbf{A}^{-1}\right| \int_{R^n} ((\mathbf{A}^{-1})_k \mathbf{y}) g(\mathbf{y}) \\ &= \left|\mathbf{A}^{-1}\right| \sum_{i=1}^{n} q_{ki} \int_{R^n} y_i g(\mathbf{y}) \end{aligned} \tag{5}$$

To solve for $\{q_{ki}\}_{i=1}^n$, more constrains must be added. Towards this goal, apply the family of left-hand compositions $\{w_p\} : R \to R$ to the known relation $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x})$, to yield

$$w_i \circ h(\mathbf{x}) = w_i \circ g(\mathbf{A}\mathbf{x}) \tag{6}$$

Integrating over both sides of the equality in (6), similarly to (5) we obtain the system

$$\begin{pmatrix} \int_{R^n} y_1 (w_1 \circ g(\mathbf{y})) & \cdots & \int_{R^n} y_n (w_1 \circ g(\mathbf{y})) \\ \vdots & \ddots & \vdots \\ \int_{R^n} y_1 (w_n \circ g(\mathbf{y})) & \cdots & \int_{R^n} y_n (w_n \circ g(\mathbf{y})) \end{pmatrix} \begin{pmatrix} q_{k1} \\ \vdots \\ q_{kn} \end{pmatrix} = \begin{pmatrix} |A| \int_{R^n} x_k (w_1 \circ h(\mathbf{x})) \\ \vdots \\ |A| \int_{R^n} x_k (w_n \circ h(\mathbf{x})) \end{pmatrix} \tag{7}$$

Similar system of equations is solved for each k to obtain the entire matrix $\mathbf{A}^{-1}$ and thus $\mathbf{A}$ itself.

We have just proved he following theorem:

Theorem 1 *Let $\mathbf{A} \in GL_n(R)$. Assume the h and g are measurable and bounded functions in $M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = g(\mathbf{Ax})$. Then given measurements of h and g, $\mathbf{A}$ can be uniquely determined if there exists a set of continuous function $\{w_p\}_{p=1}^n$ such that the matrix*

$$\begin{pmatrix} \int\limits_{R^n} y_1(w_1 \circ g(\mathbf{y})) & \cdots & \int\limits_{R^n} y_n(w_1 \circ g(\mathbf{y})) \\ \vdots & \ddots & \vdots \\ \int\limits_{R^n} y_1(w_n \circ g(\mathbf{y})) & \cdots & \int\limits_{R^n} y_n(w_n \circ g(\mathbf{y})) \end{pmatrix} \quad (8)$$

is full rank.

Remark: Note that the solution for $\mathbf{A}$ employs only zero (the Jacobian) and first order constraints (obtained by multiplying $w_p \circ h$ by x_k) and avoids the use of higher order moments. However, imposing such a restriction (which is clearly convienient due to its simplicity) may result in cases where a system of the type (8) does not exist. It is then obvious that higher order moments are needed to obtain a system similar to (8) (yet nonlinear) with enough equations to solve for all the unknowns.

5. NUMERICAL EXAMPLES

In this section we present a numerical example to illustrate the operation and performance of the proposed model, and parameter estimation algorithm.

The example illustrates the operation of the proposed algorithm on an image of a real object. The image dimensions are 1170×1750. The top image in Figure 2 depicts the original image of the aircraft, which is also employed as the template. In order to be able to evaluate the performance of the method the image of the object is then deformed to obtain the simulated deformed observation of the aircraft. See the middle image. The deforming affine transformation is given by

$$\mathbf{A} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

where the estimate obtained by the proposed procedure,

$$\hat{\mathbf{A}} = \begin{pmatrix} 0.7023 & 0.2974 \\ 0.2018 & 0.7982 \end{pmatrix} .$$

Finally, the estimated deformation is applied to the original template in order to obtain an estimate of the deformed object (lower image in Figure 2) which can be compared with the deformed object shown in the middle image.

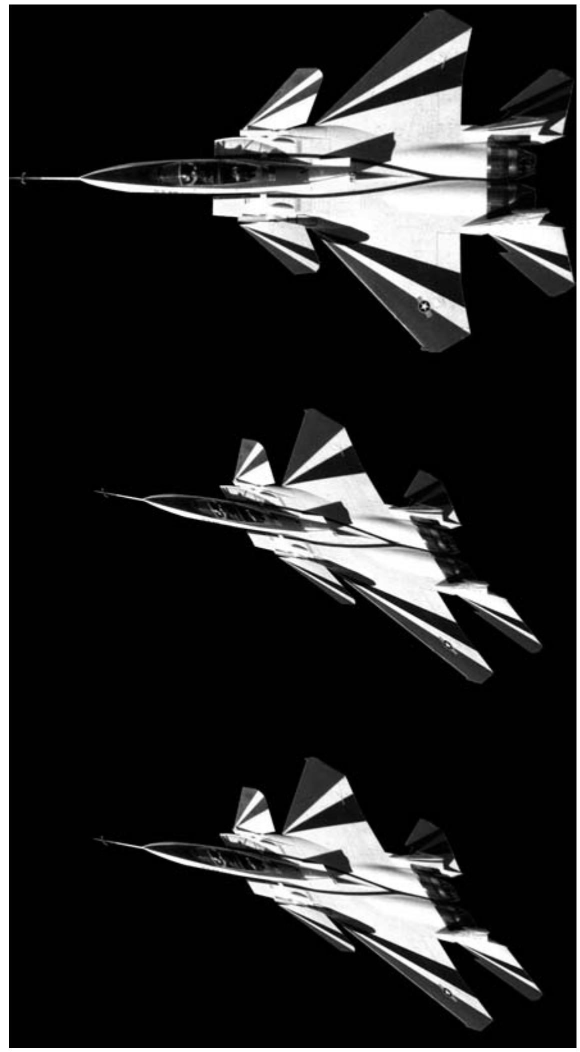

Figure 2: From top to bottom: Template; Observation on the deformed object; Estimated deformed object obtained by applying the deformation estimated from the observation to the template.

6. REFERENCES

[1] U. Grenander, *General Pattern Theory*, Oxford University Press, 1993.

[2] R. Hagege and J. M. Francos, "Parametric Estimation of Two-Dimensional Affine Transformations," submitted for publication.