Paper 3C1

Answers for Examples Sheet 3: Laplace Transform

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Laplace Transform

1. (a) We have:

$$\begin{aligned} f(t) &= e^{-\alpha t}\ \cos(\beta t + \phi) \\ &= e^{-\alpha t}\ [\cos(\beta t)\cos\phi - \sin(\beta t)\sin\phi] \end{aligned}$$

From Table of Laplace Transform Relations (TLTR):

$$\mathcal{L}\{e^{-at}[A\cos\omega_0 t + B\sin\omega_0 t]\} = \frac{A(s+a) + B\omega_0}{(s+a)^2 + \omega_0^2} \qquad (1)$$

Then

$$\mathcal{L}\{f(t)\} = \frac{\cos\phi \cdot (s+\alpha) - \sin\phi \cdot \beta}{(s+\alpha)^2 + \beta^2}$$

(b) From TLTR:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad (2)$$

and

$$\mathcal{L}\{u(t) = 1\} = \frac{1}{s} \qquad (3)$$

Then

$$\mathcal{L}\{f(t) = t^2 + 1\} = \frac{2}{s^3} + \frac{1}{s}$$

2. (a) From TLTR, We know:

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a} \qquad (4)$$

So we have to find A, B and C such as:

$$\frac{6}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \qquad (5)$$

This is done by:

- multiplying equation 5 by $(s+1)$:

$$\frac{6}{(s+2)(s+3)} = A + \frac{B(s+1)}{s+2} + \frac{C(s+1)}{s+3}$$

  and then setting $s = -1$:

$$\begin{aligned} \tfrac{6}{(-1+2)(-1+3)} &= A \\ \tfrac{6}{2} &= A \\ 3 &= A \end{aligned}$$

- multiplying equation 5 by $(s+2)$ and setting $s = -2$, we get $B = -6$.

- multiplying equation 5 by $(s+3)$ and setting $s=-3$, we get $C=3$.

Then the inverse transform is easily computed by:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\tfrac{6}{(s+1)(s+2)(s+3)}\right\} &= \mathcal{L}^{-1}\left\{\tfrac{3}{s+1}-\tfrac{6}{s+2}+\tfrac{3}{s+3}\right\}\\ &= 3\ e^{-t}-6\ e^{-2t}+3\ e^{-3t}\end{aligned}$$

(b) In this case, the decomposition is written:

$$\frac{2}{(s+1)(s^2+2^2)}=\frac{A}{(s+1)}+\frac{Bs+C}{s^2+2^2} \tag{6}$$

A, B and C are identified by:

- multiplying equation 6 by $(s+1)$ and setting $s=-1$, we get $A=\frac{2}{5}$.
- in choosing particular values $s=0$ and $s=1$ in equation 6, we get the following relations:

$$\left\{\begin{array}{l}\frac{2}{(0+1)(0+2^2)}=\frac{A}{(0+1)}+\frac{C}{(0+2^2)}\\ \frac{2}{(1+1)(1+2^2)}=\frac{A}{(1+1)}+\frac{B+C}{(1+2^2)}\end{array}\right.$$

  which imply $C=\frac{2}{5}$ and $B=\frac{-2}{5}$.

Using TLTR (cf. equations 4 and 1), the inverse transform is:

$$\mathcal{L}\left\{\frac{2}{(s+1)(s^2+2^2)}\right\}=\frac{1}{5}\left[2\ e^{-t}-2\cos 2t+\sin 2t\right]$$

(c) In this case, the decomposition is written:

$$\frac{3}{(s+1)(s+2)^2}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2} \tag{7}$$

A, B and C are identified by:

- multiplying equation 7 by $(s+1)$ and setting $s=-1$, we get $A=3$.
- multiplying equation 7 by $(s+2)^2$ and setting $s=-2$ we get $C=-3$.
- in choosing particular value $s=1$ in equation 7, we get the following relation:

$$\frac{3}{(1+1)(1+2)^2}=\frac{3}{1+1}+\frac{B}{1+2}+\frac{-9}{(1+2)^2}$$

  i.e. $B=-3$.

$$\mathcal{L}\left\{\frac{3}{(s+1)(s+2)^2}\right\}=3\ \left[e^{-t}-e^{-2t}-t\ e^{-2t}\right]$$

Transfer functions

3. (a) Poles: $\{0; -2; -8\}$ and Zeros: $\{-4\}$. We have:

$$\begin{aligned}\frac{36\ (s+4)}{s(s+2)(s+8)} &= \frac{36\ (s+2+2)}{s\ (s+2)\ (s+8)}\\ &= \frac{36}{s\ (s+8)} + \frac{72}{s\ (s+2)\ (s+8)}\end{aligned}$$

Using the methods proposed to solve the previous question, we can easily find that:

$$\begin{cases} \frac{1}{s\ (s+8)} &= \frac{\frac{1}{8}}{s} - \frac{\frac{1}{8}}{s+8}\\ \frac{1}{s\ (s+2)\ (s+8)} &= \frac{\frac{1}{16}}{s} - \frac{\frac{1}{12}}{s+2} + \frac{\frac{1}{48}}{s+8}\end{cases}$$

which implies:

$$\frac{36\ (s+4)}{s(s+2)(s+8)} = \frac{9}{s} - \frac{3}{s+8} - \frac{6}{s+2}$$

and consequently:

$$\mathcal{L}^{-1}\left[\frac{36\ (s+4)}{s(s+2)(s+8)}\right] = 9\ u(t) - 3\ e^{-8t} - 6\ e^{-2t}$$

(b) Poles: $\{0; -3; -4; -5\}$ and Zeros: $\{-1; -2\}$. We have:

$$\begin{aligned}\frac{30\ (s+1)\ (s+2)}{s\ (s+3)\ (s^2+9s+20)} &= \frac{30(s^2+3s+2)}{s\ (s+3)\ (s^2+9s+20)}\\ &= \frac{30\ s\ (s+3)+60}{s\ (s+3)\ (s^2+9s+20)}\\ &= \frac{30}{(s+5)(s+4)} + \frac{60}{s\ (s+3)\ (s+4)\ (s+5)}\end{aligned}$$

Knowing that:

$$\begin{cases} \frac{1}{(s+5)(s+4)} &= \frac{1}{s+4} - \frac{1}{s+5}\\ \frac{1}{s\ (s+3)\ (s+4)\ (s+5)} &= \frac{1}{60s} - \frac{1}{6(s+3)} + \frac{1}{4(s+4)} - \frac{1}{10(s+5)}\end{cases}$$

Hence

$$\frac{30\ (s+1)\ (s+2)}{s\ (s+3)\ (s^2+9s+20)} = \frac{-36}{s+5} + \frac{45}{s+4} + \frac{1}{s} - \frac{10}{s+3}$$

and the impulse response:

$$\mathcal{L}^{-1}\left\{\frac{30\ (s+1)\ (s+2)}{s\ (s+3)\ (s^2+9s+20)}\right\} = -36\ e^{-5t} + 45\ e^{-4t} + u(t) - 10\ e^{-3t}$$

(c) Poles: $\{0; -5\}$ and Zeros: $\{-10\}$. We have:

$$\begin{aligned}\frac{200(s+10)}{s(s+5)^2} &= 200\left[\frac{1}{s(s+5)} + \frac{5}{s(s+5)^2}\right]\\ &= 200\left[\frac{1}{5s} - \frac{1}{5(s+5)} + \frac{1}{5s} - \frac{1}{5(s+5)} - \frac{1}{(s+5)^2}\right]\\ &= \frac{80}{s} - \frac{80}{s+5} - \frac{200}{(s+5)^2}\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{200(s+10)}{s(s+5)^2}\right\} = 80\ u(t) - 80\ e^{-5t} - 200\ t\ e^{-5t}$$

(d) Poles: $\{0; -5\}$ and Zeros: $\{-3; -6\}$. We have:

$$\begin{aligned}
\tfrac{10(s^2+9s+18)}{s(s+5)} &= \tfrac{10[(s^2+5s)(1-\frac{4}{s})-2]}{s^2+5s} \\
&= 10\left(1+\tfrac{4}{s}\right) - \tfrac{20}{s(s+5)} \\
&= 10 + \tfrac{40}{s} - \tfrac{4}{s} + \tfrac{4}{s+5} \\
&= 10 + \tfrac{36}{s} + \tfrac{4}{s+5}
\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{10(s^2+9s+18)}{s(s+5)}\right\} = 10\ \delta(t) + 36\ u(t) + 4\ e^{-5t}$$

4. The amplifier is assumed to be an ideal op-amp, so the voltage at the inverting input is equal to 0. Hence the voltage drop over the R_1, C_1 section is $v_{in} - 0 = v_{in}$. The current flowing through R_1 is v_{in}/R_1 while the current flowing through C_1 is $C_1 \frac{dv_{in}(t)}{dt}$. The current flowing into the node just before the inverting input is thus

$$I_1(t) = \frac{v_{in}(t)}{R_1} + C_1 \frac{dv_{in}(t)}{dt}$$

A similar analysis shows that the current leaving the R_2, C_2 loop is

$$I_2(t) = \frac{v_{out}(t)}{R_2} + C_2 \frac{dv_{out}(t)}{dt}$$

Note that we are assuming here that $v_{out} > 0$ and thus the current $I_2(t)$ flows from right to left, meeting the current $I_1(t)$ at the node before the inverting input. Thus we can say that the total current flowing into the inverting input is equal to $I_1(t) + I_2(t)$ and equals zero. We could also assume that $v_{out} < 0$ and work with a current that flows from left to right starting from that node branch point. In this case we use the rule that the currents satisfy $I_1(t) = I_2(t)$. Either way is okay, just be consistent with your definition of voltage drops and current flows etc.

Anyway we have, (using $I_1(t) + I_2(t) = 0$)

$$\frac{v_{in}(t)}{R_1} + C_1 \frac{dv_{in}(t)}{dt} = -\frac{v_{out}(t)}{R_2} - C_2 \frac{dv_{out}(t)}{dt}$$

Take the Laplace transform of both sides, and assume that all initial conditions are zero, to yield

$$\frac{\bar{v}_{in}(s)}{R_1} + C_1 s \bar{v}_{in}(s) = -\frac{\bar{v}_{out}(s)}{R_2} - C_2 s \bar{v}_{out}(s)$$

This can be re-arranged to yield

$$\frac{\bar{v}_{out}(s)}{\bar{v}_{in}(s)} = \frac{-R_2\,(1 + sC_1R_1)}{R_1\,(1 + sC_2R_2)}$$

There are always two ways to do that kinds of questions. we can also redrawing the circuit as in figure 1 with (remember $s = j\omega$):

$$Z_1 = \frac{R_1}{1+j\omega\ R_1\ C_1} = \frac{R_1}{1+s\ R_1\ C_1}$$

$$Z_2 = \frac{R_2}{1+j\omega\ R_2\ C_2} = \frac{R_2}{1+s\ R_2\ C_2}$$

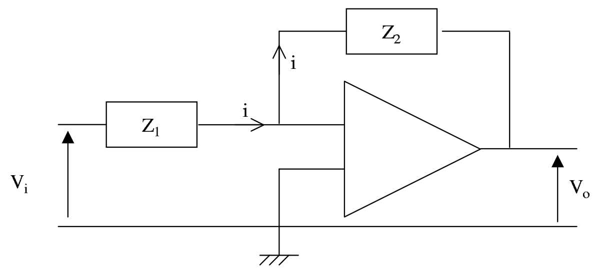


Figure 1: Circuit.

We have:

$$\begin{cases} V_o = -Z_2\ i \\ V_i = Z_1\ i \end{cases}$$

implying that:

$$\frac{V_o}{V_i} = \frac{R_2\ (1+s\ R_1\ C_1)}{R_1\ (1+s\ R_2\ C_2)}$$

5. By introducing an unknown signal E, the transfer functions of the block diagrams can be found as indicated figure 2.

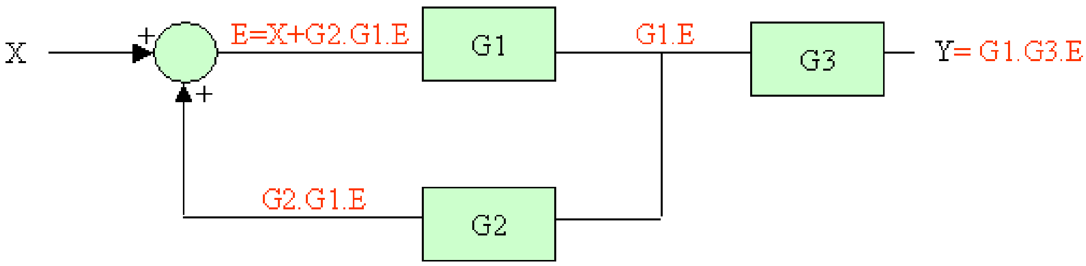
Impulse and Step responses

6. We have $V_{out} = Z\ I_{in}$ with $(s = j\omega)$:

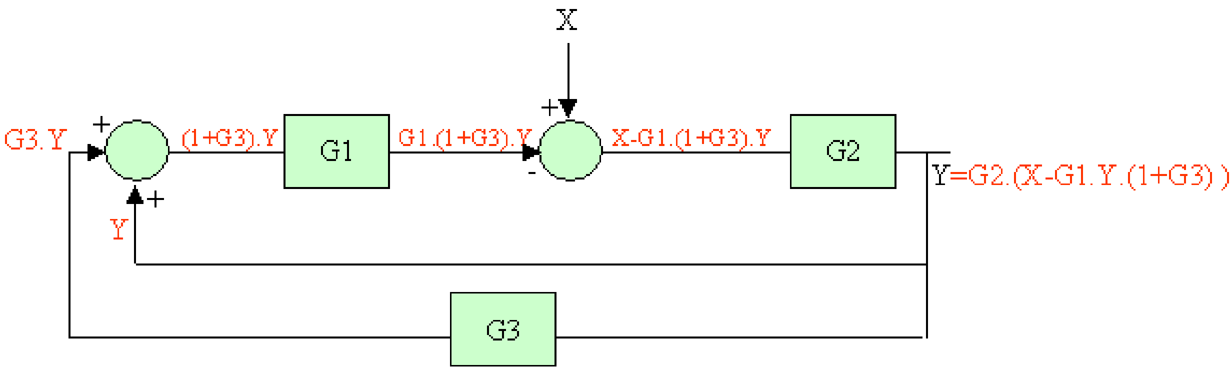
$$\begin{aligned} Z &= \left[\frac{1}{R+Ls} + Cs\right]^{-1} \\ &= \frac{R+Ls}{1+Cs(R+Ls)} \end{aligned}$$

The impulse response is:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{10(s+6)}{s^2+6s+10}\right\} &= \mathcal{L}^{-1}\left\{\frac{10(s+3)}{(s+3)^2+1} + \frac{30}{(s+3)^2+1}\right\} \\ &= e^{-3t}[10\ \cos(t) + 30\ \sin(t)] \end{aligned}$$



(a): $X(s) = (1 - G1(s)G2(s))\ E(s)$ and $Y(s) = G3(s)G1(s)E(s)$. Hence $\frac{Y(s)}{X(s)} = \frac{G3(s)\ G1(s)}{1-G1(s)G2(s)}$.



(b): $\frac{Y(s)}{X(s)} = \frac{G2(s)}{1+G1(s)G2(s)(1+G3(s))}$.

Figure 2: Block Diagrams solutions to question 5.

This can be rewritten using the relation $\alpha\cos(t-B) = \alpha\cos(t)\cos(B) - \alpha\sin(A)\sin(t)$, and then solving $\alpha\cos(A) = 10$ and $\alpha\sin(A) = 30$ or:

$$\left|\begin{array}{l} \tan(A) = 3 \\ \alpha = \frac{10}{\cos(A)} \end{array}\right.$$

The impulse response is then rewritten as (using radians): $e^{-3t} \cdot 31.62 \cdot \cos(t - 1.25)$.

The step response is:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{10(s+6)}{s(s^2+6s+10)}\right\} &= \mathcal{L}^{-1}\left\{\frac{10}{(s^2+6s+10)} + \frac{60}{s(s^2+6s+10)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{10}{(s^2+6s+10)} + \frac{6}{s} - \frac{6s+36}{(s^2+6s+10)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{10}{((s+3)^2+1)} + \frac{6}{s} - \frac{6(s+3)}{((s+3)^2+1)} - \frac{18}{((s+3)^2+1)}\right\} \\
&= 6\ u(t) - e^{-3t}\ [6\ \cos(t) - 8\ \sin(t)]
\end{aligned}$$

The step response can be rewritten as (using radians): $6 + 10\ e^{-3t} \cdot \cos(t + 2.21)$.

7. What does the information in the question mean? Well, it means that we have to solve two problems, the first with the switch in position 1, and the second with the switch in position 2. The initial conditions for position 1 are assumed to be zero, while the final state of the system for the first problem becomes the initial conditions for the second problem.

   Let's tackle the first problem with the switch at position 1. The 40 Volt source is turned on at $t = 0$. The source term is thus $40u(t)$ and the governing equation for the current is

$$40u(t) = 2i(t) + \frac{1}{4}\frac{di}{dt}$$

   with initial condition $i(0) = 0$. Taking the Laplace transform, and using the fact that the initial conditions are zero current, we get

$$\frac{40}{s} = 2\bar{i}(s) + \frac{1}{4}s\bar{i}(s)$$

   Rearranging, we get that

$$\begin{aligned} \bar{i}(s) &= \frac{160}{s(8+s)} \\ &= 160\left(\frac{A}{s} + \frac{B}{s+8}\right) \\ &= 160\left(\frac{1}{8s} - \frac{1}{8(s+8)}\right) \\ &= 20\frac{1}{s} - 20\frac{1}{s+8} \end{aligned}$$

   Inverting the transform we get that

$$i(t) = 20u(t) - 20e^{-8t}$$

   The question says that the circuit is left in this state for a while implying that by the time the switch is moved to position 2 the transient $20e^{-8t}$ has died away to nearly zero, and the current is a constant 20 Amps. Hence when the switch is moved (at some new time $t = 0$), removing the voltage source, we have the new equation for $i(t)$

$$0 = 2i(t) + \frac{1}{4}\frac{di}{dt}$$

   with *new initial condition* $i(0) = 20$. Applying the Laplace transform we get

$$0 = 2\bar{i}(s) + \frac{1}{4}\left(s\bar{i}(s) - i(0)\right)$$

   or

$$\bar{i}(s) = \frac{20}{s+8}$$

   yielding

$$i(t) = 20e^{-8t}$$

8. The Laplace transform of the equation is (assuming zero initial conditions):

$$s^2\ Y(s) + 2\ \alpha\ s\ Y(s) + (\Omega^2 + \beta^2)\ Y(s) = X(s)$$

Hence, the transfer function is:

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + 2\alpha s + \Omega^2 + \beta^2}$$

(a) $\beta = \alpha$ then:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+\alpha)^2 + \Omega^2}$$

Hence from TLTR (cf. eq. 1):

$$h(t) = \frac{e^{-\alpha t}\sin(\Omega t)}{\Omega}$$

(b) $\beta = \alpha$ and $\Omega^2 = -A^2$:

$$H(s) = \frac{1}{(s+\alpha)^2 - A^2}$$

Hence from TLTR:

$$h(t) = \frac{e^{-\alpha t}\sinh(At)}{A}$$

(c) $\Omega^2 + \beta^2 = \alpha^2$

$$H(s) = \frac{1}{(s+\alpha)^2} = -\frac{d}{ds}\left(\frac{1}{s+\alpha}\right)$$

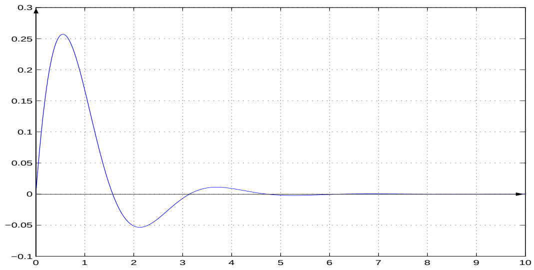
Hence from TLTR:

$$h(t) = t\ e^{-\alpha t}$$

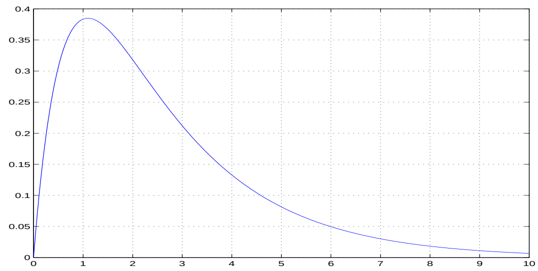
(d) equivalent to (a):

$$h(t) = \frac{e^{-\alpha t}\sin(\Omega t)}{\Omega}$$

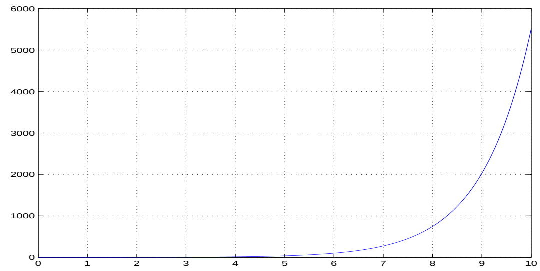
remark: since $\alpha < 0$ then $|\alpha| = -\alpha$.



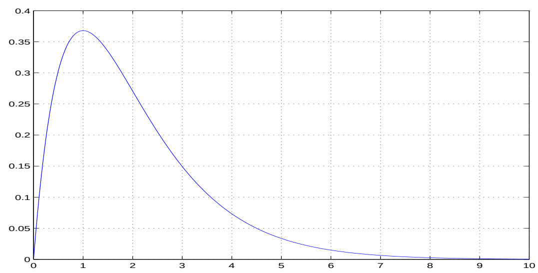
(a): $h(t) = \frac{e^{-\alpha t}\sin(\Omega t)}{\Omega}$ with $\alpha = 1 > 0,\ \Omega = 2$



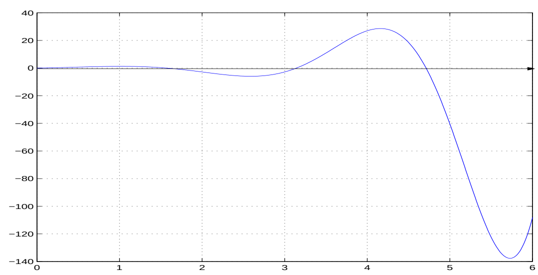
(b): $h(t) = \frac{e^{-\alpha t}\sinh(At)}{A}$ with $\alpha = 1,\ A = 0.5$



(b): $h(t) = \frac{e^{-\alpha t}\sinh(At)}{A}$ with $\alpha = 1,\ A = 2$



(c): $h(t) = t\ e^{-\alpha t}$ with $\alpha = 1 > 0$



(d): $h(t) = \frac{e^{-\alpha t}\sin(\Omega t)}{\Omega}$ with $\alpha = -1,\ \Omega = 2$ (unstable)

Figure 3: Impulse responses for question 8.