Paper 3C1

Answers for Examples Sheet 4: Convolution

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1. The convolution, $C(t)$, is defined by

$$C(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\ x_2(\tau)\ d\tau$$

We are given that $x_2(\tau) \equiv 0$ for $\tau < 0$. We can thus change the lower limit on the integral and get

$$C(t) = \int_{0}^{\infty} x_1(t-\tau)\ x_2(\tau)\ d\tau$$

Similarly $x_1(t-\tau) = 0$ for $t-\tau < 0$ or $\tau > t$. Hence we can change the upper limit on the integral and get

$$C(t) = \int_{0}^{t} x_1(t-\tau)\ x_2(\tau)\ d\tau$$

or

$$\begin{aligned} C(t) &= \int_0^t e^{-a(t-\tau)}\ e^{-b\tau}\ d\tau \\ &= e^{-at} \int_0^t e^{\tau(a-b)}\ d\tau \end{aligned}$$

If $a = b$ we get

$$\begin{aligned} C(t) &= e^{-at} \int_0^t 1\ d\tau \\ &= te^{-at} \end{aligned}$$

If $a \neq b$ we get

$$\begin{aligned} C(t) &= e^{-at} \left. \frac{e^{\tau(a-b)}}{a-b} \right|_0^t \\ &= e^{-at} \left(\frac{e^{t(a-b)} - 1}{a-b} \right) \\ &= \frac{e^{-bt} - e^{-at}}{a-b} \end{aligned}$$

2. The received (output) signal is given by the convolution of the input signal and impulse response. This is a very important standard result that we covered in notes. For this example we get

$$y(t) = x(t) * h(t)$$

Using the results from question one (as, apart from a factor of β, the input signal and impulse response have the same form as x_1 and x_2 in that question) we get for $\alpha \neq \beta$

$$y(t) = \beta \left(\frac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta} \right)$$

while for $\alpha = \beta$

$$y(t) = \beta t e^{-\beta t}$$

Repeating the question involves using the standard result

$$\mathcal{L}\{x(t) * h(t)\} = \mathcal{L}\{x(t)\}\,\mathcal{L}\{h(t)\}$$

Hence we have that

$$\begin{aligned}\mathcal{L}\{y(t)\} &= \mathcal{L}\{x(t) * h(t)\} \\ &= \mathcal{L}\{x(t)\}\,\mathcal{L}\{h(t)\}\end{aligned}$$

and so

$$y(t) = \mathcal{L}^{-1}\{\mathcal{L}\{x(t)\}\,\mathcal{L}\{h(t)\}\}$$

The Laplace transform of the input signal is

$$\mathcal{L}\{x(t)\} = \frac{1}{s+\alpha}$$

while the Laplace transform of the impulse response is given by

$$\mathcal{L}\{h(t)\} = \frac{\beta}{s+\beta}$$

The output is thus

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{(s+\alpha)\,(s+\beta)}\right\}$$

Now if $\alpha \neq \beta$ we get (using partial fractions)

$$y(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{\alpha-\beta}\cdot\frac{1}{s+\beta} + \frac{\beta}{\beta-\alpha}\cdot\frac{1}{s+\alpha}\right\}$$

Taking inverse Laplace transforms yields, as required

$$y(t) = \beta\left(\frac{e^{-\beta t} - e^{-\alpha t}}{\alpha-\beta}\right)$$

Meanwhile if $\alpha = \beta$ we have, as required

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left\{\frac{\beta}{(s+\beta)^2}\right\} \\ &= \beta t e^{-\beta t}\end{aligned}$$

(using the Laplace transform of t and the shift theorem.)

3. Remember from handout that stability is governed by the location of the poles of the transfer function. Poles are values of s that lead to undefined (infinite) values for the transfer function, that is they lead to a zero value in the denominator. The rules are:
   - If all the poles are in the left side of the complex plane, that is, all poles have negative real

parts, the system is asymptotically stable.
• A pole in the right hand complex plane, that is, a pole with a positive real part, implies that the system is unstable.
• A distinct pole on the imaginary axis, along with all other poles in the left side of the complex plane, implies that the system is marginally stable.
• A pole on the imaginary axis with mulitiplicity greater than 1, implies that the system is unstable.
Armed with these facts we can easily see that

(a) Pole at $s = -3$, therefore stable.

(b) Pole at $s = -3$, therefore stable.

(c) Pole at $s = 3$, therefore unstable.

(d) Pole of multiplicity 1 at $s = 0$ and no other poles, therefore marginally stable.

(e) Pole of multiplicity 2 at $s = 0$, therefore unstable.

(f) Poles of multiplicity 1 at $s = \sqrt{8}\jmath$ and $s = -\sqrt{8}\jmath$, therefore marginally stable.

(g) Poles at $s = \sqrt{8}$ and $s = -\sqrt{8}$, therefore unstable.

(h) Poles at $s = -5$, $s = \frac{1 \pm \sqrt{3}\jmath}{2}$ therefore unstable.

(i) Poles at $s = -5$, $s = \frac{-1 \pm \sqrt{3}\jmath}{2}$ therefore stable.

4. This question is tricky because responses 1,3,4 look practically the same. But it is solvable.

First of all note that response 5 is the only unstable response ... it grows exponentially with time and does not decay. On the left hand side, system D is the only one with poles on the right hand side of the s-plane, hence that is the only unstable system. Thus response 5 comes from system D.

Of the other systems, only C has its poles on the real axis, yet three responses 1,3,4 have well damped behaviour. Now the closer poles are to the real axis, the more undamped the impulse response becomes. Therefore as system A has its poles closest to the real axis thus it is the most undamped system. The response 2 is the only one with oscillations, so A must have yielded impulse response 2.

Now we are left with responses 1,3,4 to sort out.

If you look on the left hand side, systems B and E have identical poles except E is scaled by a factor of 30 compared to B. Thus system E should be 30 times MORE DAMPED than than B. Thus the impulse response of E should be something like 30 times FASTER than B. On the right hand side impulse response 4 is about 30 times faster than response 3. Thus system E yields impulse response 4 while B yields impulse response 3.

Then we are left with just system C and response 1. So C must have yielded impulse response 1.