Paper 3C1

Answers for Examples Sheet 6: Fourier Analysis

Anil Kokaram and Rozenn Dahyot 2003
http://www.mee.tcd.ie/~sigmedia

1. Using the complex form of the fourier series we get

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\jmath k\omega_0 t}$$

where

$$\begin{aligned} a_k &= \frac{1}{T}\int_0^T x(t)e^{-\jmath k\omega_0 t}\,dt \\ &= \frac{E}{T}\int_0^T e^{-t\left(\jmath k\omega_0+\frac{5}{T}\right)}\,dt \\ &= \frac{-E}{T}\left.\frac{e^{-t\left(\jmath k\omega_0+\frac{5}{T}\right)}}{\frac{5}{T}+\jmath k\omega_0}\right|_0^T \\ &= \frac{E}{T\left(\frac{5}{T}+\jmath k\omega_0\right)}\left(1-e^{-T\left(\frac{5}{T}+\jmath k\omega_0\right)}\right) \end{aligned}$$

The d.c. term is a_0 or

$$a_0 = \frac{E}{5}\left(1-e^{-5}\right) = .199E \text{ Volts}$$

The fundamental corresponds to a frequency of $\omega_0 = \frac{2\pi}{T}$ radians per second. We have

$$\begin{aligned} a_1 &= \frac{E}{T\left(\frac{5}{T}+\jmath\frac{2\pi}{T}\right)}\left(1-e^{-T\left(\frac{5}{T}+\jmath\frac{2\pi}{T}\right)}\right) \\ &= \frac{E}{(5+\jmath 2\pi)}\left(1-e^{-(5+\jmath 2\pi)}\right) \\ &= \frac{E}{(5+\jmath 2\pi)}\left(1-e^{-5}\right) \end{aligned}$$

or

$$a_1 = (0.0770 - \jmath 0.0968)E$$

The periodic signal can be written $x(t) = a_0 + a_1 \exp^{(\omega_0 t)} + a_1^* \exp^{(-\omega_0 t)} + \cdots$ and then the amplitude of the fundamentale is $2\times|a_1| = 2*\sqrt{0.0770^2+0.0968^2}E = 0.2474E$. It can be shown that the transfer function of the system is

$$H(s) = \frac{1}{1+sRC}$$

The frequency response is thus given by

$$H(\jmath\omega) = \frac{1}{1+\jmath\omega RC}$$

The d.c. gain is thus given by

$$H(0) = \frac{1}{1+0} = 1$$

The phase gain is

$$\phi = \arg\frac{1}{1} = \arg 1 = 0$$

So the d.c. component is unaffected by the filter.

For the fundamental frequency $\omega = \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{RC}$, the gain is:

$$H(\jmath\omega_0) = \frac{1}{1+\jmath 2\pi}$$

Therefore $|H(\jmath\omega_0)| = \frac{1}{\sqrt{1+(2\pi)^2}} = 0.1572$, and the amplitude at the filter output is $|H(\jmath\omega_0)| * 0.2474E = 0.0389E$.

2.

$$\begin{aligned} X(\omega) * Y(\omega) &= \int_{-\infty}^{+\infty} X(\omega_0)\ Y(\omega-\omega_0)\ d\omega_0 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(\omega_0)\ y(t)\ e^{-\jmath(\omega-\omega_0)t}\ dt\ d\omega_0 \\ &= \int_{-\infty}^{+\infty} y(t)\ e^{-\jmath\omega t}\left(\int_{-\infty}^{+\infty} X(\omega_0)\ e^{\jmath\omega_0 t}\ d\omega_0\right) dt \\ &= \int_{-\infty}^{+\infty} y(t)\ e^{-\jmath\omega t}\ x(t)\ dt \\ &= \mathrm{TF}\{y(t)\cdot x(t)\} \end{aligned}$$

3. a)

$$\begin{aligned} F(\omega) &= \int_0^{\infty} 10e^{-t(3+\jmath\omega)}\ dt \\ &= \left.\frac{10e^{-t(3+\jmath\omega)}}{3+\jmath\omega}\right|_{\infty}^{0} \\ &= \frac{10}{3+\jmath\omega} \end{aligned}$$

b)

$$\begin{aligned} F(\omega) &= \int_0^{\infty} te^{-t(1+\jmath\omega)}\ dt \\ &= \left.\frac{te^{-t(1+\jmath\omega)}}{1+\jmath\omega}\right|_{\infty}^{0} + \int_0^{\infty}\frac{e^{-t(1+\jmath\omega)}}{1+\jmath\omega}\ dt \\ &= \left.\frac{e^{-t(1+\jmath\omega)}}{(1+\jmath\omega)^2}\right|_{\infty}^{0} \\ &= \frac{1}{(1+\jmath\omega)^2} \end{aligned}$$

c)

$$\begin{aligned} F(\omega) &= \int_0^\infty e^{-t(2+\jmath\omega)}\ dt \\ &= \left.\frac{e^{-t(2+\jmath\omega)}}{2+\jmath\omega}\right|_\infty^0 \\ &= \frac{1}{2+\jmath\omega} \end{aligned}$$

d)

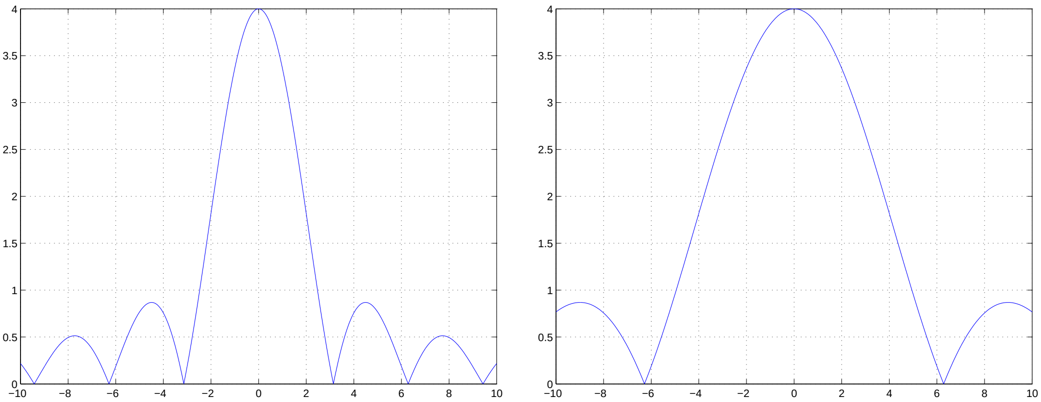
$$\begin{aligned} F(\omega) &= \int_{-1}^{1} 2\ e^{-\jmath\omega t}\ dt \\ &= \left.\frac{-2\ e^{-\jmath\omega t}}{\jmath\omega}\right|_{-1}^{1} \\ &= \frac{-2}{\jmath\omega}\left(e^{-\jmath\omega} - e^{\jmath\omega}\right) \\ &= \frac{4\sin(\omega)}{\jmath\omega} \end{aligned}$$

e)

$$\begin{aligned} F(\omega) &= \int_{-1/2}^{1/2} 2\ e^{-\jmath\omega t}\ dt \\ &= \left.\frac{-2e^{-\jmath\omega t}}{\jmath\omega}\right|_{1/2}^{-1/2} \\ &= \frac{4\sin(\frac{\omega}{2})}{\jmath\frac{\omega}{2}} \end{aligned}$$

According to figure 1, the narrower the pulse the wider the bandwidth.

4. a)With $\omega_0 = \frac{2\pi}{T}$, $e^{\jmath\omega_0\frac{T}{4}} = \jmath$ and $e^{-\jmath\omega_0\frac{T}{4}} = -\jmath$, the Fourier transform $X(\omega)$ of the signal $x(t)$ is



(d) (e)

Figure 1: Plots of (d) $|F(\omega)| = \left|\frac{4\sin(\omega)}{\omega}\right|$ and (e) $|F(\omega)| = \left|\frac{4\sin(\frac{\omega}{2})}{\frac{\omega}{2}}\right|$ in question 3.

computed by:

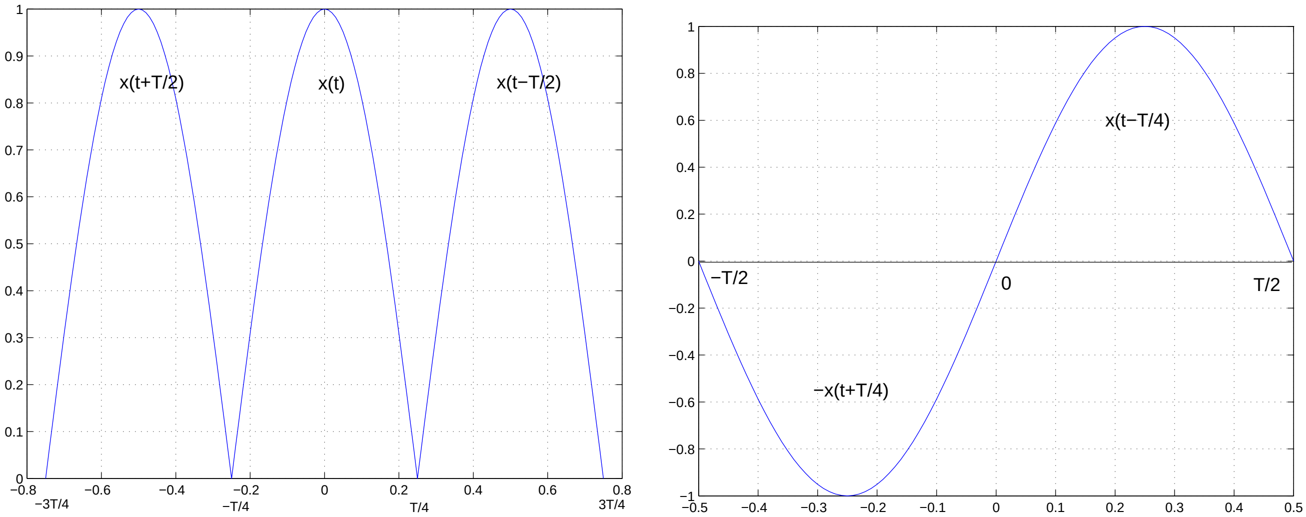
$$\begin{aligned}
\int_{-T/4}^{T/4} \cos(\omega_0 t)\ e^{-\jmath\omega t}\ dt &= \int_{-T/4}^{T/4} \frac{e^{-\jmath\omega_0 t}+e^{-\jmath\omega_0 t}}{2}\ e^{-\jmath\omega t}\ dt \\
&= \frac{1}{2}\left[\frac{e^{\jmath(\omega_0-\omega)t}}{\jmath(\omega_0-\omega)} - \frac{e^{\jmath(\omega_0+\omega)t}}{\jmath(\omega_0+\omega)}\right]_{-T/4}^{T/4} \\
&= \frac{1}{2\jmath(\omega_0^2-\omega^2)}\left[(\omega_0+\omega)e^{\jmath(\omega_0-\omega)t} - (\omega_0-\omega)e^{\jmath(\omega_0+\omega)t}\right]_{-T/4}^{T/4} \\
&= \frac{1}{2\jmath(\omega_0^2-\omega^2)}\left[(\omega_0+\omega)(\jmath\ e^{-\jmath\omega\frac{T}{4}} + \jmath\ e^{+\jmath\omega\frac{T}{4}}) + (\omega_0-\omega)(\jmath\ e^{-\jmath\omega\frac{T}{4}} + \jmath\ e^{+\jmath\omega\frac{T}{4}})\right] \\
&= \frac{1}{2(\omega_0^2-\omega^2)}\left[\omega_0(e^{-\jmath\omega\frac{T}{4}} +\ e^{+\jmath\omega\frac{T}{4}})\right] \\
&= \frac{2\omega_0\cos(\omega\frac{T}{4})}{(\omega_0^2-\omega^2)}
\end{aligned}$$

b) The signal is $y(t) = x(t+T/2) + x(t) + x(t-T/2)$ implying $Y(\omega) = X(\omega)(e^{\jmath\omega\frac{T}{2}} + 1 + e^{-\jmath\omega\frac{T}{2}})$ or:

$$\begin{aligned}
Y(\omega) &= X(\omega)\,[1 + 2\cos(\omega T/2)] \\
&= \\
&= \frac{2\omega_0}{\omega_0^2-\omega^2}[\cos(\omega T/4) + 2\cos(\omega T/4)\cos(\omega T/2)]
\end{aligned}$$

Remember $\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$ so $2\cos(\omega T/4)\cos(\omega T/2) = \cos(3\omega T/4) + \cos(\omega T/4)$. Finally:

$$Y(\omega) = \frac{2\omega_0}{\omega_0^2-\omega^2}[2\cos(\omega T/4) + \cos(3\omega T/4)]$$



$y(t) = x(t+T/2) + x(t) + x(t-T/2)$ $\qquad$ $y(t) = x(t-T/4) - x(t+T/4)$

Figure 2: Functions $y(t)$ for (b) and (c) in question 4.

c) The signal is $y(t) = x(t - T/4) - x(t + T/4)$ implying $Y(\omega) = X(\omega)(e^{-\jmath\omega\frac{T}{4}} - e^{\jmath\omega\frac{T}{4}})$ or:

$$\begin{aligned} Y(\omega) &= X(\omega)\ (-2\jmath)\sin(\omega T/4) \\ &= \tfrac{-2\jmath\omega_0}{\omega_0^2-\omega^2}[2\cos(\omega T/4)\sin(\omega T/4)] \end{aligned}$$

Knowing that $2\cos(a)\sin(a) = \sin(2a)$, we have:

$$Y(\omega) = \frac{-2\jmath\omega_0}{\omega_0^2 - \omega^2}\sin(\omega T/2)$$