Paper 3C1

Answers for Examples Sheet 7: Digital Systems

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1. Nyquist theorem stipulates that to preserve the information content of an analogue signal after sampling, the sampling rate must be at least twice the maximum frequency present in the signal. If $f_s < 2f_{max}$ then aliasing will occur.

   All of these signals, (a) to (d), are real signals. Therefore, even though their maximum frequency content is given, there will always be some residual content due to noise,for instance, above these frequencies.

   The anti-aliasing filter will pass this noise as well as the signal if the passband is chosen as $= f_{max}$. This is because there is a non-zero transition band attenuation. Therefore, if the bandwidth of the original signal is $\pm B_x$ Hz then the effective bandwidth after anti-aliasing (using a low pass filter with pass band width $\pm B_x$ Hz) is $\pm 1.2 B_x$. Therefore the minimum sampling frequency must be $2.4 B_x$. So (a) 240 Hz, (b) $3400 \times 2.4 = 8160$ Hz, (c) 16kHz $\times 2.4 = 38.4$ kHz, (d) 5.5MHz $\times 2.4 = 13.2$ MHz.

2. (a) Figure 1 shows a sketch of the signal $s(t)$.

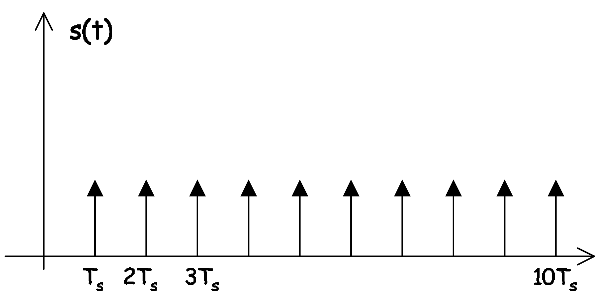


Figure 1: (a) sketch of $s(t)$ for question 2.

The Fourier Series expansion of $s(t) = \sum_{k=-\infty}^{+\infty} a_k \ e^{j\omega_s k t}$ is such that:

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) \ e^{j\omega_s k t} \ dt \\ &= \frac{1}{T_s} \end{aligned} \quad (1)$$

Hence

$$s(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} e^{j\omega_s k t}$$

(b)

$$\begin{aligned} X_s(\omega) &= \int_{-\infty}^{+\infty} x_s(t)\ e^{j\omega t}\ dt \\ &= \int_{-\infty}^{+\infty} \left[\frac{1}{T_s}\sum_{k=-\infty}^{+\infty} e^{j\omega_s k t}\right] x(t)\ e^{j\omega t}\ dt \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-j(\omega-\omega_s k)t}\ x(t)\ dt \\ &= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X(\omega - k\omega_s) \end{aligned}$$

where $\omega_s = 2\pi f_s$.

(c) Using $f_s = 2.25B_x$ it is possible to recover $x(t)$ from $x_s(t)$ provided:

(i) anti-alias filter with $(H(f)\ )$ transition band of $\leq 12.5\%$ of pass-band is used.

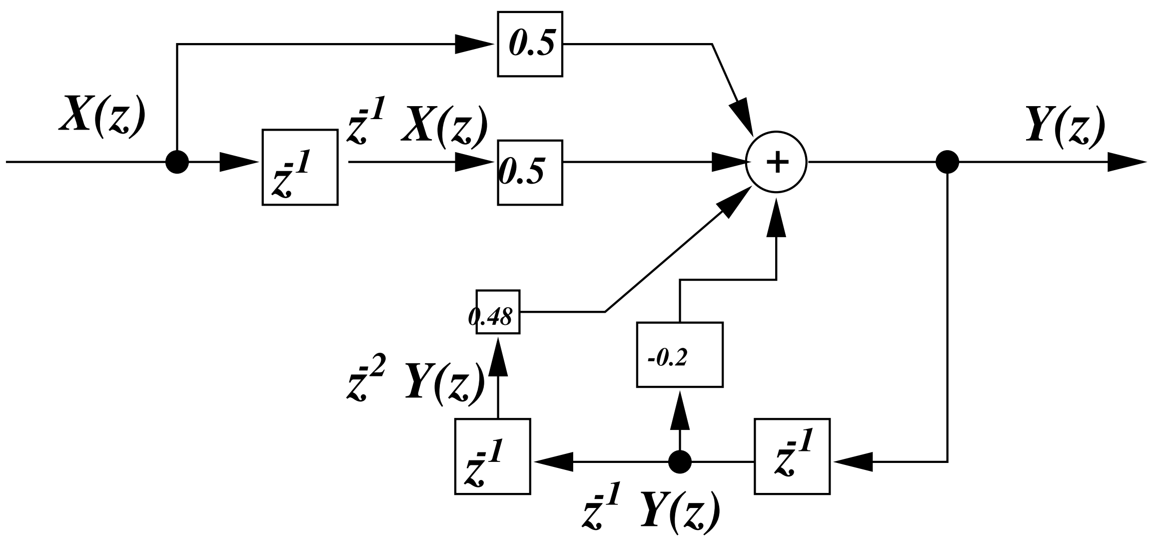
(ii) reconstruction filter $|H_r(f)|$ must have cut-off at $f = 1.125B$. May need boost at high frequencies to compensate for sample/ operation due to A to D conversion.

3. This question follows almost exactly exercises done in lectures.

   (a) To draw the block diagram, write the system transfer function in terms of the Z-Transform of an input and an output signal, $\mathbf{X}(z)$ and $\mathbf{Y}(z)$ respectively. This makes it easier to see the relationships between blocks.

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = \frac{0.5(1+z^{-1})}{1+0.2z^{-1}-0.48z^{-2}}$$
$$\Rightarrow \mathbf{Y}(z)[1+0.2z^{-1}-0.48z^{-2}] = \mathbf{X}(z)[0.5(1+z^{-1})]$$
$$\Rightarrow \mathbf{Y}(z) = \mathbf{X}(z)[0.5(1+z^{-1})] - \mathbf{Y}(z)[0.2z^{-1}-0.48z^{-2}]$$
$$\Rightarrow \mathbf{Y}(z) = 0.5\mathbf{X}(z) + 0.5\mathbf{X}(z)z^{-1} - 0.2\mathbf{Y}(z)z^{-1} + 0.48\mathbf{Y}(z)z^{-2}$$

   Hence the block diagram is as below.



(b) Given the system transfer function $\mathbf{H}(z)$, the impulse response is the inverse Z-Transform of $\mathbf{H}(z)$. To find this inverse transform, use partial fractions in this case.

$$\begin{aligned}\mathbf{H}(z) &= \frac{0.5(1+z^{-1})}{1+0.2z^{-1}-0.48z^{-2}} \\ &= \frac{0.5(1+z^{-1})}{(1-0.6z^{-1})(1+0.8z^{-1})} \\ &= \frac{A}{1-0.6z^{-1}} + \frac{B}{1+0.8z^{-1}}\end{aligned}$$

Using cover up for A, set $z^{-1} = 10/6$ and cover up

$$A = \frac{(1+(10/6))/2}{(1+(8/6))} = \frac{4}{7}$$

Using cover up for B, set $z^{-1} = -10/8$ and cover up

$$\begin{aligned}B &= \frac{(1-(10/8))/2}{1-(6/8)} = \frac{-1}{14} \\ \Rightarrow \mathbf{H}(z) &= \frac{(4/7)}{1-0.6z^{-1}} + \frac{(9/14)}{1+0.8z^{-1}}\end{aligned}$$

The impulse response h_n is $\mathcal{Z}^{-1}(\mathbf{H}(z))$ hence, from tables

$$h_n = \mathcal{Z}^{-1}(\mathbf{H}(z)) = (4/7)(0.6)^n + (-1/14)(-0.8)^n$$

(c) To find the first four terms of the step response there are three approaches. Since the question only askes about the first 4 terms, we can convert the system transfer function into a difference equation, input a step and turn the handle. The second method is to calculate the step response of the system using Z-transforms i.e. taking the inverse Z-Transform of $\frac{1}{1-z^{-1}}\mathbf{H}(z)$. The last method, and perhaps the fastest, is to use the relationship that the step response is the running sum of the impulse response.

i. Using the fact that the step response is the running sum of the impulse response, we have

$$\begin{aligned}u_n &= \sum_{k=0}^{n} h_k \\ &= \sum_{k=0}^{n} (4/7)(0.6)^k + (-1/14)(-0.8)^k\end{aligned}$$

Thus calculating for $n = 0,\ 1,\ 2,\ 3$ we get

$$
\begin{aligned}
u_0 &= \sum_{k=0}^{0}(4/7)(0.6)^k + (-1/14)(-0.8)^k \\
&= (4/7)(0.6)^0 + (-1/14)(0.8)^0 = (4/7) + (-1/14) = 0.5 \\
u_1 &= \sum_{k=0}^{1}(4/7)(0.6)^k + (-1/14)(-0.8)^k \\
&= (4/7)(0.6)^0 + (-1/14)(0.8)^0 + (4/7)(0.6)^1 + (-1/14)(-0.8)^1 \\
&= 0.5 + (4/7)(0.6) + (-1/14)(-0.8) = 0.9
\end{aligned}
$$

And so on for $n = 2,\ 3$

$$
\begin{aligned}
u_2 &= 1.06 \\
u_3 &= 1.22
\end{aligned}
$$

ii. Alternative: Using the difference equation directly. From the expression that we used for drawing the block diagram, the difference equation can be derived. Remember that z^{-1} is a shift in time of one sample hence the inverse z xform of $\mathbf{X}(z)z^{-1}$ is x_{n-1} and so on for z^{-2} etc.

$$\mathbf{Y}(z) = 0.5\mathbf{X}(z) + 0.5\mathbf{X}(z)z^{-1} - 0.2\mathbf{Y}(z)z^{-1} + 0.48\mathbf{Y}(z)z^{-2}$$

Taking inverse Z-Transforms to get the difference equation

$$y_n = 0.5x_n + 0.5x_{n-1} - 0.2y_{n-1} + 0.48y_{n-2}$$

Now, we input a step i.e. $x_n = 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots$ (rembering that the step function is 0 for ALL $n < 0$ and 1 otherwise) and use your calculator

$$
\begin{aligned}
y_0 &= 0.5x_0 + 0.5x_{0-1} - 0.2y_{0-1} + 0.48y_{0-2} \\
&= 0.5 + 0 - 0 + 0 = 0.5 \qquad (2) \\
y_1 &= 0.5x_1 + 0.5x_{1-1} - 0.2y_{1-1} + 0.48y_{1-2} \\
&= 0.5x_1 + 0.5x_0 - 0.2y_{-1} + 0.48y_{-2} \\
&= 0.5 + 0.5 - 0.2 \times 0.5 + 0.48 \times 0 \\
y_2 &= .9
\end{aligned}
$$

etc.

(3)

iii. Alternative: Calculating an expression for the step response using Z-Transforms

$$
\begin{aligned}
\frac{\mathbf{H}(z)}{1-z^{-1}} &= \frac{0.5(1+z^{-1})}{(1-0.6z^{-1})(1+0.8z^{-1})(1-z^{-1})} \\
&= \frac{A}{(1-0.6z^{-1})} + \frac{B}{1+0.8z^{-1}} + \frac{C}{1-z^{-1}}
\end{aligned}
$$

Get A using cover up rule and set $z^{-1} = 10/6$

$$A = -6/7$$

Get B using cover up rule and set $z^{-1} = -10/8$

$$B = -2/63$$

Get C using cover up rule and set $z^{-1} = 1$

$$C = 25/18 \quad \rightarrow \mathbf{H}(z)\frac{1}{1-z^{-1}} = \frac{(-6/7)}{(1-0.6z^{-1})} + \frac{(-2/63)}{1+0.8z^{-1}} + \frac{(25/18)}{1-z^{-1}}$$
$$\rightarrow \mathcal{Z}^{-1}(\mathbf{H}(z)\frac{1}{1-z^{-1}}) = g_n = (-6/7)(0.6)^n - (2/63)(-0.8)^n + (25/18)$$

From which the first four terms of the step response are had by setting $n = 0,\ 1,\ 2,\ 3$ in the expression to yield $g_n = 0.5,\ 0.9,\ 1.06,\ 1.22$ For the first 4 terms as required

The first method of answering this part of the question is WAY easier in this case. But I have put all the three methods so you can use them if you want to revise those others.

4. This question is almost exclusively about difference equations. The first part of the question exercises your knowledge about the connection between difference equations and the Z-Transform.

(a) Given the system transfer function $\mathbf{H}(z)$, to convert back to a difference equation just remember that z^{-1} is like a one-sample memory buffer. Therefore $z^{-1}\mathbf{X}(z)$ is the Z-Transform of x_{n-1} and so on. Also remember that the system transfer function is a summary of the input/output relationship for a dynamic system. Hence $\mathbf{H}(z) = \mathbf{Y}(z)/\mathbf{X}(z)$ where $\mathbf{Y},\ \mathbf{X}$ are the Z-Transforms of the output and input sequences into the system respectively. Hence:

$$\mathbf{H}(z) = \frac{\mathbf{Y}(z)}{\mathbf{X}(z)} = 1 + 2z^{-1} + z^{-2}$$
$$\Rightarrow \mathbf{Y}(z) = \mathbf{X}(z)[1 + 2z^{-1} + z^{-2}]$$

Taking inverse Z-transforms we can write down the difference equation directly as follows

$$y_n = x_n + 2x_{n-1} + x_{n-2}$$

(b) We are given an input sequence starting at $n = 0$ as $x_n = 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0,$ and zero thereafter. To calculate the output from the given system when presented with this input, we can use the difference equation above directly. The alternative is to use Z-transforms i.e. calculate the Z-Transform of the input signal to get $\mathbf{X}(z)$ then multiply by the transfer function and take the inverse Z-Transform to get $y_n = \mathcal{Z}^{-1}(\mathbf{X}(z)\mathbf{H}(z))$. Taking this latter approach is definitely possible but way too long winded for this simple problem. So we can proceed as follows ...

$$
\begin{aligned}
y_0 &= x_0 + 2x_{0-1} + x_{0-2} \\
&= x_0 + 2x_{-1} + x_{-2} \\
&= 0 + 0 + 0 = 0 \\
y_1 &= x_1 + 2x_{1-1} + x_{1-2} \\
&= x_1 + 2x_0 + x_{-1} \\
&= 0 + 0 + 0 = 0 \\
y_2 &= x_2 + 2x_{2-1} + x_{2-2} \\
&= x_2 + 2x_1 + x_0 \\
&= 1 + 0 + 0 = 1 \\
y_3 &= x_3 + 2x_{3-1} + x_{3-2} \\
&= x_3 + 2x_2 + x_1 \\
&= 1 + 2 + 0 = 3
\end{aligned}
$$

and so on to get

$$
\begin{aligned}
y_4 &= 4 \\
y_5 &= 4 \\
y_6 &= 4 \\
y_7 &= 3 \\
y_8 &= 1 \\
y_9 &= 0
\end{aligned}
$$

y_n is zero thereafter.

(c) To answer this part of the question it helps to plot the input and output signals and have a look (cf. fig. 2). You will see first of all that the output signal looks sortof like the input signal but for three things

i) the signal shape is a "blurred" pulse. The input signal is a pulse of height 1 and length 5 samples. But the output is a trapezoid of length 7 samples and height 4 samples. The system has *spread* the input shape. ii) The DC gain of this system is 4. That is, if I put a signal that is completely FLAT and of amplitude 1, the output of this system (in steady state) is another signal that is also completely flat, but of height 4. You can see this yourself by plugging a flat signal through the difference equation. So the other change that the system introduces to the input signal is that it multiplies it by a factor of 4. iii) The output signal y_n lags the input by 2 samples. The maximum of the output is reached 2 samples after the rising edge of the pulse. This makes sense since the filter is CAUSAL and has a maximum lag in its difference equation of 2 samples i.e. x_{n-2}

The system is clearly FIR: a Finite Impulse Response filter. This is because it operates only on current and past inputs. Therefore its transfer function $\mathbf{H}(z)$ has NO POLES! Another way to prove this is to work out the impulse response of the filter (which is going to be 1, 2, 1) and show that it is of FINITE duration.

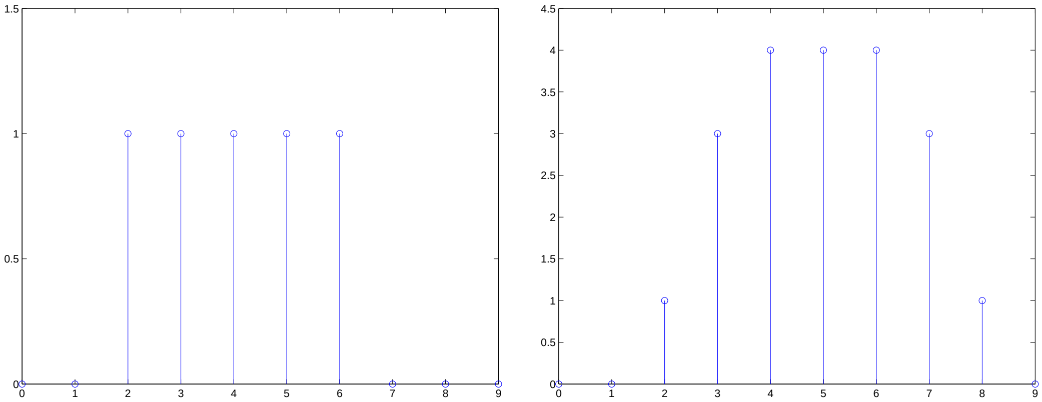


Figure 2: Input and output signals.