SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 2.
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SYSTEM RESPONSE

- Ultimate goal is to control a process or analyze a signal To do this need to be able to predict the response of a system to various inputs
- Given known system dynamics (i.e. we know enough about the system to write down ODE's describing its operation) we can use the Laplace Transform to work out the response of the system to *any* input whose Laplace Transform can be found.
- However, real inputs are not predictable; and it may not be possible to write down a system's ODE's. The process used in Mobile Phones for canceling echoes is a good example of a system which operates despite the inability to analytically define the geometry/dynamics of the phone's immediate environment.
- Still can identify some basic system characteristics which allow handling of system response. This is possible using purely time domain analysis.
- Time domain analysis relies on the fact that the response of a system to an impulse or to a step tells you everything about the system (in principle). These responses can be calculated if the ODE's for the system can be written, or they can be measured by experiment beforehand or on-line (e.g. mobile phones, noise cancellation etc.)
- We will explore both techniques for analyzing systems. Start with Laplace analysis as you met this in 2nd year.

1 Review of Laplace Analysis

We'll be using the Laplace Transform to solve differential equations. Need to be confident in using it.

1. Laplace Transform of a signal $f(t)$ is
$$\mathcal{L}\Big(f(t)\Big) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$
s is a COMPLEX number, e.g. $\sigma + j\omega$!! The result of taking the Laplace Transform of a signal $f(t)$ is a function in s. So we go from a 1-dimensional function i.e. a function of time only, to a function of 2 variables σ and ω i.e. a function of a *complex variable*. We will write the laplace Transform of $f(t)$ as $\mathbf{F}(s)$. Sometimes in books you *might* see $\bar{f}(s)$ whic is useful because some capital letters look like the common ones when written by hand e.g. V and v.

2. Inverse laplace Transform of $\mathbf{F}(s)$ is
$$f(t) = \mathcal{L}^{-1}\Big(\mathbf{F}(s)\Big) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \mathbf{F}(s)e^{st}ds \tag{2}$$
This is a nasty[1] integral to do, need to know about Contour integration. Happily, being Engineers we can use tables of Laplace Transform pairs to do this.

3. Sometimes people panic about the convergence of the inverse Laplace Transform, because you find yourself having to say, for instance
$$\int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s} \tag{3}$$

[1] Not really, but it makes you feel better if you think this is so.

even though you don't know what value s is. Don't panic, in this course we are going to assume that the Laplace integral always converges (i.e. Re(s) > 0) and so

$$\left[-\frac{e^{-st}}{s}\right]\Bigg|_{s=\infty} = 0 \tag{4}$$

BUT YOU NEED TO REMEMBER THAT CONVERGENCE IS AN ISSUE

4. In addition, assume that $e^{-(s+a)t}$ is 0 for $s = \infty$, (i.e. you can always assume that $s + a > 0$ to make this happen.)

5. Remember to spot the 'shift' theorem since it gives a slick way of using tables to get Laplace transforms of functions (we'll derive this later for the Fourier transform as well).

$$\text{Given } \mathcal{L}\Big(f(t)\Big) = \mathbf{F}(s)$$
$$\text{Then } \mathcal{L}\Big(e^{-kt}f(t)\Big) = \mathbf{F}(s+k)$$

In other words, **if you want to find the Laplace transform of $e^{-kt}f(t)$, then just find the Laplace transform of $f(t)$ and replace all occurrences of s with $s + k$.** A similar statement can be made for the inverse transform.

1.1 TABLE OF LAPLACE TRANSFORM RELATIONS

Waveform: $g(t)$ (defined for $t \geq 0$)	Laplace Transform: $\mathbf{G}(s) = \mathcal{L}\{g(t)\} = \int_{0_-}^{\infty} g(t)e^{-st}dt$
$\delta(t)$ impulse	1
$u(t)$ unit step	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_0^2}$
$\sinh(\omega_0 t)$	$\frac{\omega_0}{s^2-\omega_0^2}$
$\cosh(\omega_0 t)$	$\frac{s}{s^2-\omega_0^2}$
$e^{-at}[A\cos(\omega_0 t) + B\sin(\omega_0 t)]$	$\frac{A(s+a)+B\omega_0}{(s+a)^2+\omega_0^2}$
$e^{-at}g(t)$	$\mathbf{G}(s+a)$ shift in s
$g(t-\tau)u(t-\tau)$ where $\tau \geq 0$	$e^{-s\tau}\mathbf{G}(s)$ shift in t
$tg(t)$	$-\frac{d}{ds}\mathbf{G}(s)$
$\frac{dg}{dt}$ differentiation	$s\mathbf{G}(s) - g(0)$
$\frac{d^2g}{dt^2}$ 2nd differential	$s^2\mathbf{G}(s) - sg(0) - \left(\frac{dg}{dt}\right)\Big\|_0$
$\frac{d^ng}{dt^n}$	$s^n\mathbf{G}(s) - s^{n-1}g(0) - s^{n-2}\left(\frac{dg}{dt}\right)\Big\|_0 - \ldots - \left(\frac{d^{n-1}g}{dt^{t-1}}\right)\Big\|_0$
$\int_0^t g(\tau)d\tau$ integration	$\frac{\mathbf{G}(s)}{s}$
$g_1(t) * g_2(t)$ convolution $= \int_0^t g_1(t-\tau)g_2(\tau)d\tau$	$\mathbf{G}_1(s)\mathbf{G}_2(s)$

1.2 Laplace Transform Examples

1. Find the Laplace Transform of e^{-at} from first principles

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}dt$$

$$=$$

$$=$$

$$=$$

$$= \frac{1}{s+a}$$

2. Find the Laplace Transform of $\delta(t-a)$ from first principles

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st}dt$$

$$=$$

3. Find the Laplace Transform of $\delta(t-a)$ from first principles

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}dt$$

$$=$$

4. Laplace transform of $e^{-at}+e^{-bt}$. Laplace transform obeys superposition so..

$$\mathcal{L}\{e^{-at}+e^{-bt}\} =$$

$$=$$

5. Find the Laplace transform of $e^{-at}\sin(\beta t+\phi)$
from first principles (i.e. the hard way) :

$$\mathcal{L}\left\{e^{-at}\sin(\beta t+\phi)\right\} = \int_0^\infty e^{-at}\sin(\beta t+\phi)e^{-st}dt$$

Using $\sin(x) = \frac{1}{2j}\left(e^{jx}-e^{-jx}\right)$

$$\begin{aligned}
&= \int_0^\infty e^{-at}\frac{1}{2j}\left(e^{j(\beta t+\phi)}-e^{-j(\beta t+\phi)}\right)e^{-st}dt \\
&= \frac{1}{2j}\int_0^\infty e^{-(s+a)t}\left(e^{j(\beta t+\phi)}-e^{-j(\beta t+\phi)}\right)dt \\
&= \frac{1}{2j}\int_0^\infty \left(e^{j\phi}e^{-(s+a-j\beta)t}-e^{-j\phi}e^{-(s+a+j\beta)t}\right)dt \\
&= \frac{1}{2j}\left(e^{j\phi}\left[\frac{e^{-(s+a-j\beta)t}}{-(s+a-j\beta)}\right]_0^\infty - e^{-j\phi}\left[\frac{e^{-(s+a+j\beta)t}}{-(s+a+j\beta)}\right]_0^\infty\right) \\
&= \frac{1}{2j}\left(\frac{e^{j\phi}}{(s+a-j\beta)}-\frac{e^{-j\phi}}{(s+a+j\beta)}\right) \\
&= \frac{1}{2j}\left(\frac{(s+a+j\beta)e^{j\phi}-(s+a-j\beta)e^{-j\phi}}{(s+a)^2+\beta^2}\right) \\
&= \frac{1}{2j}\left(\frac{(s+a)(e^{j\phi}-e^{-j\phi})+j\beta(e^{j\phi}-e^{-j\phi})}{(s+a)^2+\beta^2}\right) \\
&= \frac{(s+a)\sin(\phi)}{(s+a)^2+\beta^2}+\frac{\beta\cos(\phi)}{(s+a)^2+\beta^2}
\end{aligned}$$

But using tables,

$$
\begin{aligned}
\sin(\omega_0 t) &\leftrightarrow \frac{\omega_0}{s^2+\omega_0^2} \\
\cos(\omega_0 t) &\leftrightarrow \frac{s}{s^2+\omega_0^2} \\
e^{-at}\sin(\omega_0 t) &\leftrightarrow \text{Using the shift theorem: Replace } s \text{ with } s+a \\
&\leftrightarrow \frac{\omega_0}{(s+a)^2+\omega_0^2}
\end{aligned}
$$

we can do the same thing the slick way (i.e. from tables and spotting the shift theorem) :

$$
\begin{aligned}
&\mathcal{L}\Big\{e^{-at}\sin(\beta t+\phi)\Big\} = \\
&\qquad = \mathcal{L}\Big\{e^{-at}[\sin(\beta t)\cos(\phi)+\cos(\beta t)\sin(\phi)]\Big\} \\
&\qquad = \mathcal{L}\Big\{ \qquad\qquad \Big\} + \mathcal{L}\Big\{e^{-at} \qquad\qquad \Big\} \\
&\qquad = \cos(\phi)\mathcal{L}\Big\{ \qquad\qquad \Big\} + \sin(\phi)\mathcal{L}\Big\{ \qquad\qquad \Big\} \\
&\qquad = \frac{\beta\cos(\phi)}{(s+a)^2+\beta^2} + \frac{(s+a)\sin(\phi)}{(s+a)^2+\beta^2}
\end{aligned}
$$

1.3 Review of partial fractions

You will need to find the inverse Laplace Transform of sometimes complicates expressions. The method of partial fractions allows you to do this by expressing a complicated fraction in terms of a sum of simpler fractions. The idea is that the inverse laplace transform of the simpler fractions is easier to spot.

Basic idea

$$\frac{k_3 s + 1}{s(s+k_1)(s+k_2)} = \frac{A}{s} + \frac{B}{s+k_1} + \frac{C}{s+k_2} \tag{1}$$

Expand r.h.s. and equate coeffs

$$= \frac{A(s+k_1)(s+k_2) + Bs(s+k_2) + Cs(s+k_1)}{s(s+k_1)(s+k_2)}$$

$$\begin{aligned}
\text{Equating coeffs in } s^2 \Rightarrow & 0 = A + B + C \\
\text{Equating coeffs in } s \Rightarrow & k_3 = A(k_1 + k_2) + Bk_2 + Ck_1 \\
\text{Equating constants} \Rightarrow & 1 = Ak_1k_2
\end{aligned}$$

Then solve simultaneous equations for $A,\ B, C$. Urrgh, could be a pain. There's another, easier way (cover up rule)

Multiply eqn. 1 by s, set $s = 0$, This gives A straightaway

$$\Rightarrow \frac{k_3 s + 1}{(s+k_1)(s+k_2)} = \frac{As}{s} + \frac{Bs}{s+k_1} + \frac{Cs}{s+k_2} \tag{2}$$

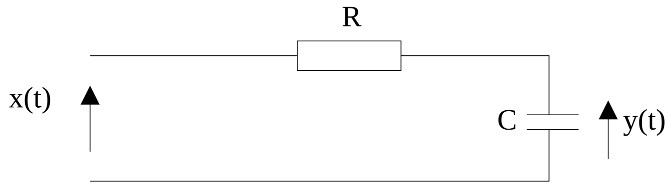
$$s = 0 \Rightarrow \frac{1}{k_1k_2} = A$$

Rule is *Set s to be value that makes a denominator factor equal 0. Then COVER UP that factor in denominator and subsitute to get coefficient value.* To get C then ...

$$s = -k_2 \Rightarrow \frac{-k_3k_2 + 1}{-k_2(k_1 - k_2)} = C \tag{3}$$

2 System Response (at last)

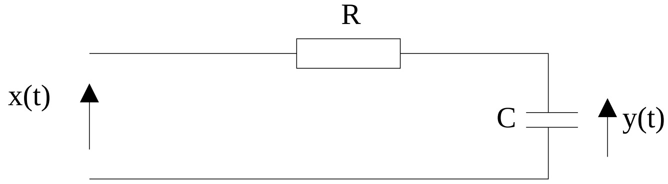
- What is the response of the system shown to a unit impulse input at $x(t)$, i.e. What happens to $y(t)$ when $x(t) = \delta(t)$ Volts ?



- Steps are
  1. Write down the differential equations for modeling the system.
  2. Given the stated initial conditions (if any) get an expression for $y(t)$ in terms of $x(t)$. In other words : solve the differential equations.
  3. Do this by
     (a) First principles using P.I. and Homogeneous solution. (A pain).
     (b) Laplace Transforms (much easier).
  4. Insert the numerical values or expression for the input $x(t)$ as well as initial conditions.
  5. Solution falls out.

2.1 Example 1 :

Impulse response of this system is the response $y(t)$ when $x(t) = \delta(t)$.



Write down differential equations to model the system.

$$x(t) = Ri(t) + y(t) = RC\frac{\mathrm{d}y}{\mathrm{dt}} + y(t)$$

Take Laplace Transforms, and solve for $\mathbf{Y}(s)$

$$\begin{aligned} \mathbf{X}(s) &= RC(s\mathbf{Y}(s) - y(0)) + \mathbf{Y}(s) \\ \mathbf{Y}(s)\Big(1 + RCs\Big) &= \mathbf{X}(s) - RCy(0) \\ \Rightarrow \mathbf{Y}(s) &= \frac{\mathbf{X}(s) - RCy(0)}{1 + RCs} \end{aligned} \tag{4}$$

Assuming initially, output is zero i.e. for $t < 0$, $y(t) = 0$

$$= \qquad \mathbf{X}(s) = G(s)X(s) \tag{5}$$

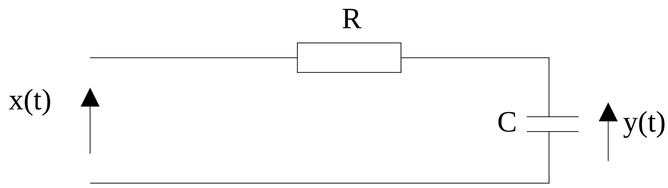
Find Laplace Transform of input $x(t)$: $\mathbf{X}(s) = \mathcal{L}\{\delta(t)\} = 1$

Hence the impulse response, (we'll refer to it as $h(t)$) is therefore

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\Big[G(s)X(s)\Big] = \mathcal{L}^{-1}\Big[G(s) \times 1\Big] \\ &= \mathcal{L}^{-1}\left[\frac{1}{1 + RCs}\right] = \mathcal{L}^{-1}\left[\frac{1/(RCs)}{(1/(RCs) + s}\right] \\ &= \end{aligned} \tag{6}$$

2.2 Step response

Impulse response of this system is the response $y(t)$ when $x(t) = u(t)$ (i.e. x(t) is a step function.



$$\Rightarrow \mathbf{Y}(s) = \frac{1}{1 + RCs}\mathbf{X}(s) \tag{7}$$

Find Laplace Transform of input $x(t)$: $\mathcal{L}\{u(t)\} = \frac{1}{s}$

Hence the step response $y(t)$ is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\mathbf{Y}(s)\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{1+RCs}\mathbf{X}(s)\right] \\ &= \mathcal{L}^{-1}\left[\left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right)\right] \end{aligned}$$

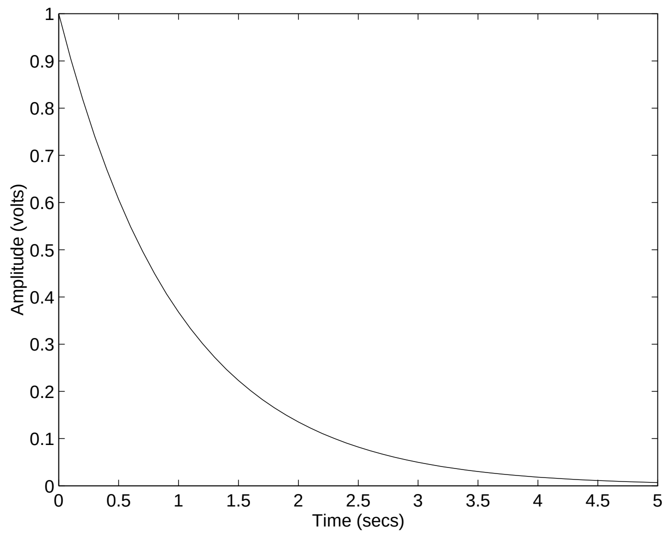
$$\text{use partial fractions:} \quad \frac{1}{s(1+RCs)} = \frac{A}{s} + \frac{B}{1+RCs}$$

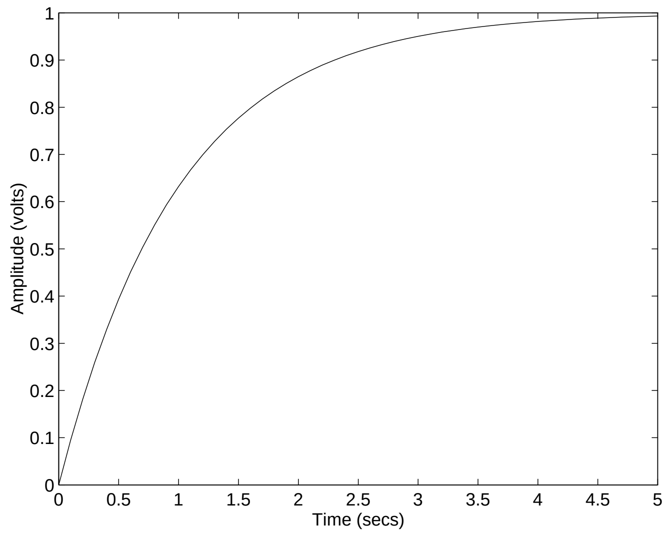
Use Cover up rule to find coefficients A, B

$$\begin{aligned} \text{Hence } \; y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{RC}{1+RCs}\right] \\ &= \\ &= u(t) - e^{-t/RC} \\ &= \end{aligned}$$

2.3 PLOTS OF THE STEP AND IMPULSE RESPONSE

(assuming zero initial conditions, R=10KΩ, C = 10μF.)





3 A time domain relationship

- The step function is the integral of the delta function.

$$u(t) = \int_0^t \delta(\tau) d\tau \tag{8}$$

- All systems we deal with are LTI so SUPERPOSITION APPLIES
  So if input x_i gives output y_i then $\sum_i x_i$ gives output
  .

- Hence

$$\text{Input} \int_0^t x(\tau) d\tau \rightarrow \text{ Output} \tag{9}$$

- So input $u(t)$ (which is the integral of $\delta(t)$) gives output

- Therefore *The step response of a system is the integral of the impulse response.*

- Now do the example 1 again, but this time *integrate* the impulse response to get the step response.

4 TRANSFER FUNCTIONS

Recall for the capacitor and resistor circuit we had

$$\begin{aligned} \mathbf{Y}(s) &= \frac{\mathbf{X}(s) - RCy(0)}{1 + RCs} \\ &= \frac{\mathbf{X}(s)}{1 + RCs} - \frac{RCy(0)}{1 + RCs} \\ \Rightarrow \mathbf{Y}(s) &= F(s)\mathbf{X}(s) + G(s)y(0) \end{aligned} \tag{10}$$

$F(s)$ is called the transfer function of the system. Its all we need to work out everything about the system if the initial conditions were 0. The Transfer function *generalizes* the idea of 'Gain' to dynamic attributes of the system.

LAPLACE TRANSFORM OF OUTPUT SIGNAL = LAPLACE TRANSFORM OF INPUT SIGNAL X SYSTEM TRANSFER FUNCTION

\+ OTHER TERMS (INITIAL CONDITIONS)

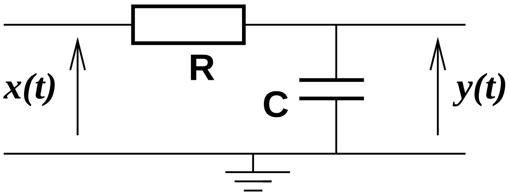
So the system block diagram can be drawn as follows.

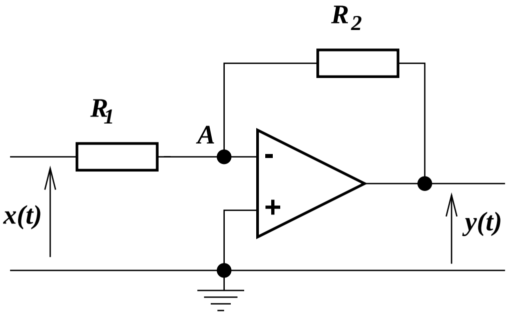
5 A link between *Impulse response* and *Transfer Function*

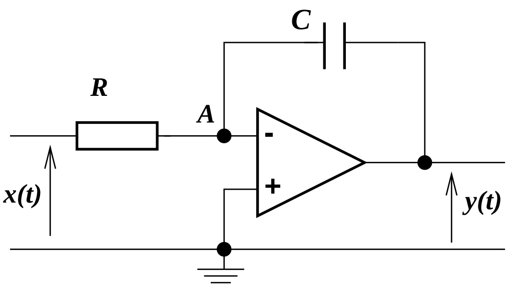
Suppose Impulse response of system is $h(t)$. What is its Transfer Function?

6 Transfer Function examples

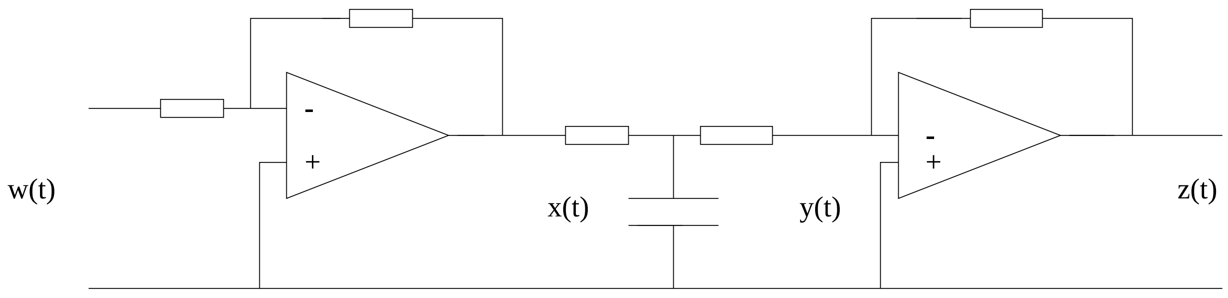
$$\text{Transfer Function } \mathbf{G}(s) = \frac{\text{Laplace Transform of Output Signal}}{\text{Laplace Transform of Input signal}} = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)}$$







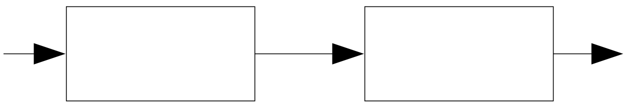
6.1 A cascade of electrical systems

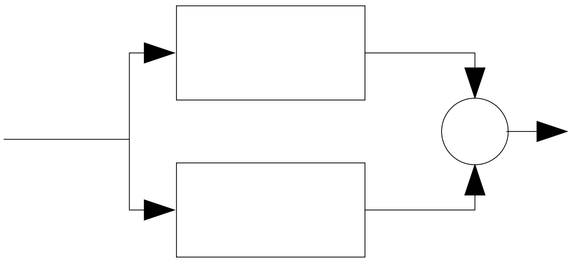


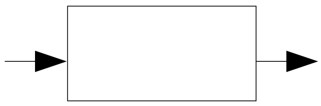
Assuming that each stage does not load the preceeding one (i.e. each input impedance >> output impedance of previous stage), Then

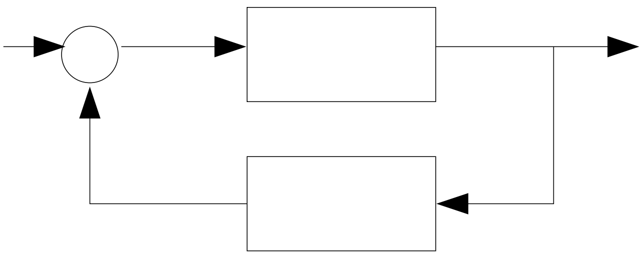
$$\frac{\mathbf{X}(s)}{\mathbf{w}(s)} = -k_1; \qquad \frac{\mathbf{Y}(s)}{\mathbf{X}(s)} = \frac{1}{1+sT}; \qquad \frac{\mathbf{z}(s)}{\mathbf{Y}(s)} = -k_2$$

7 Block Diagram Algebra

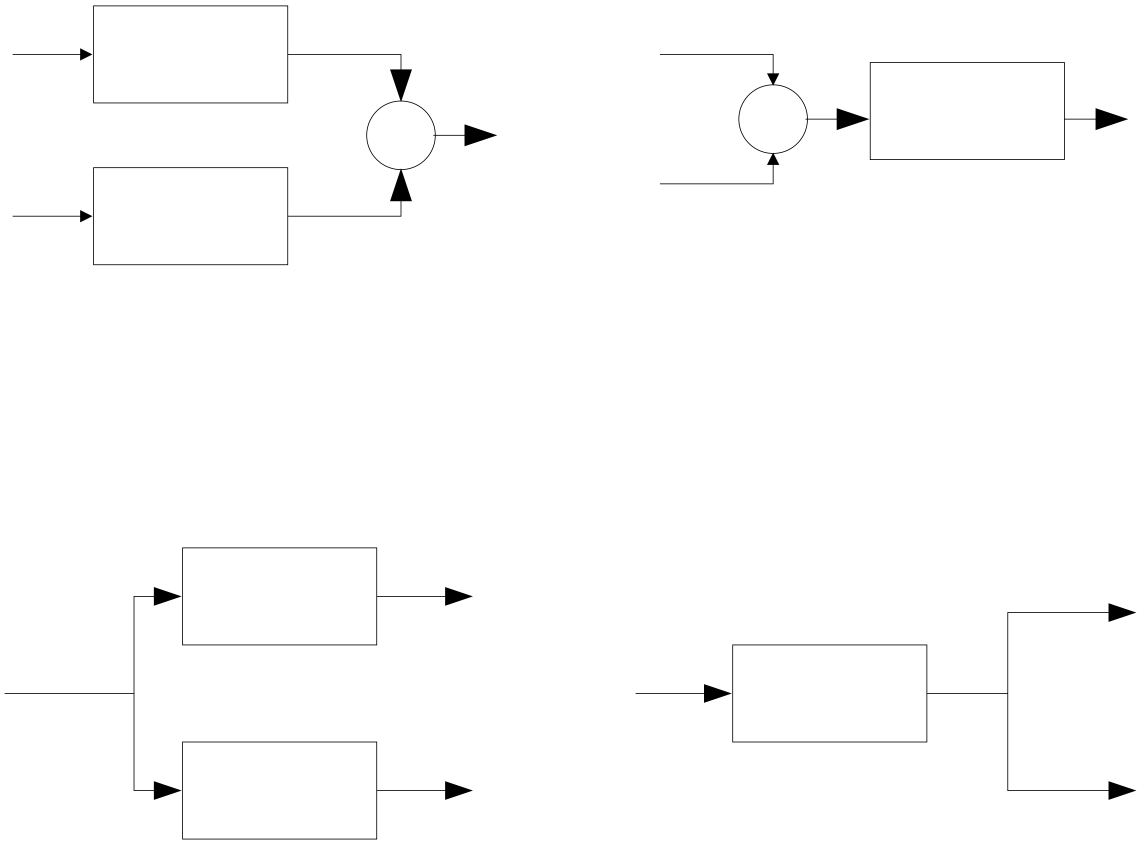








8 More Block Diagram Algebra



EXAMPLES OF THE DISTINCTION BETWEEN TRANSFORMS OF SIGNALS AND **TRANSFER FUNCTIONS** OF **SYSTEMS**.

SIGNALS	**SYSTEMS**
$1 \leftrightarrow \delta(t)$	$\mathbf{Y}(s) = 1 \cdot \mathbf{X}(s) \quad \leftrightarrow \quad y(t) = x(t)$
$\frac{1}{s} \leftrightarrow u(t)$	$\mathbf{Y}(s) = \frac{1}{s} \cdot \mathbf{X}(s) \quad \leftrightarrow \quad y(t) = \int_0^t x(\tau) d\tau$
$\frac{1}{s+a} \leftrightarrow e^{-at}$	$\mathbf{Y}(s) = \frac{1}{s+a} \cdot \mathbf{X}(s) \quad \leftrightarrow \quad \dot{y}(t) + ay(t) = x(t)$
$\frac{\omega}{s^2+\omega^2} \leftrightarrow \sin(\omega t)$	$\mathbf{Y}(s) = \frac{\omega}{s^2+\omega^2} \cdot \mathbf{X}(s) \quad \leftrightarrow \quad \ddot{y}(t) + \omega^2 y(t) = \omega x(t)$
$e^{-sT} \leftrightarrow \delta(t-T)$	$\mathbf{Y}(s) = e^{-sT} \cdot \mathbf{X}(s) \quad \leftrightarrow \quad y(t) = x(t-T)$

Transfer functions arise through Laplace Analysis of **Differential Equations** which represent the dynamic behaviour of **systems**. They summarise the input/output behaviour of a **system** in the s domain (i.e. they generalize the concept of 'gain'). Finding the Laplace *transform* of a signal is a step one takes when it is required to find out what a particular system (represented by its transfer function) does to an input signal.

It just turns out that the Laplace *transform* of some signals have a form which occurs also as a Transfer function of particular **systems**.

Terminology (will be used later in examining stability)

Suppose

$$\mathbf{G}(s) = \frac{n(s)}{d(s)} \tag{11}$$

Then the roots of $n(s)$ are called the ZEROS of the system $\mathbf{G}(s)$

And the roots of $d(s)$ are called the POLES of the system $\mathbf{G}(s)$

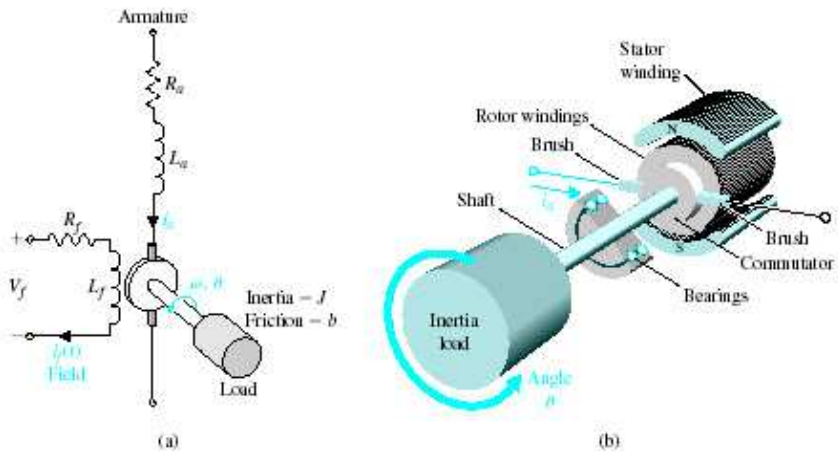
EXAMPLE:

$$\begin{aligned} G(s) &= \frac{4s^2 - 8s - 60}{s^3 + 2s^2 + 2s} \\ &= \frac{4(s+3)(s-5)}{s(s+1+j)(s+1-j)} \end{aligned}$$

Zeros of $G(s)$ are the values of s that make the numerator $= 0$ hence

Poles of $G(s)$ are the values of s that make the denominator $= 0$ hence

8.1 Another Example: FIELD CONTROLLED DC MOTOR



Left: Circuit Diagram, Right: The Motor

DC Motors found in Hard Drives, CD Players, Scalextric Model car sets

How does $\theta(t)$ (the rotational angle of the axle) depend on $e(t)$ (the voltage applied to the field winding)? The following system equations apply:

Linearised Motor Equation (k=some const.)

$$\tau(t) = ki(t) \tag{1}$$

Torque = Inertia × acceleration

$$\tau(t) - B\dot{\theta}(t) = J\ddot{\theta}(t) \tag{2}$$

Kirchoff

$$e(t) = Ri(t) + L\frac{\partial i}{\partial t} \tag{3}$$

1. What is the the system transfer function $\frac{\theta(s)}{\mathbf{E}(s)}$?
2. What is the impulse response of the system?
3. What is the response of the system to a unit step impulse at $t = 0$?

Assume $\theta(0) = 0,\ \dot{\theta}(t) = 0,\ i(0) = 0$; First have a think about what you expect ...

Now some analysis:

- To get the transfer function we need to express the Laplace transform of the output signal $\tau(s)$ (torque) as a function of the laplace transform of the input signal $\mathbf{I}(s)$ (current). So take Laplace xforms of the differential equations representing the dynamic behaviour of each system (remember tables!) ...

  From eqn 1

  $$= \tag{4}$$

  From eqn. 2

  $$\Rightarrow \tau(s) - \quad =$$

  But from initial conditions, $\theta(0) = 0,\ \dot{\theta}(t) = 0,\ i(0) = 0$ so

  $$\tau(s) = \mathbf{\Theta}(s)[Bs + Js^2] \tag{5}$$

  From eqn. 3

  $$\mathbf{E}(s) = \tag{6}$$

  $$= \mathbf{I}(s)[R + Ls]$$

Let $T_f = L/R,\ T_m = J/B$ (to reduce the amount of writing to do). Hence from eqn. 6

$$\mathbf{I}(s) = \frac{\mathbf{E}(s)}{R(1 + sT_f)} \tag{7}$$

Still have some work to do ... don't panic ... all we want to do is to get $\mathbf{\Theta}(s) = \text{somefunctionof}(\mathbf{E}(s))$

Subst $\mathbf{I}$ from eqn 7 into 4

$$\Rightarrow \tau(s) = \tag{8}$$

Subst τ from 8 into 5

$$= \mathbf{\Theta}(s)[Bs + BT_m s^2]$$

$$= \mathbf{\Theta}(s)Bs[1 + T_m s]$$

So

$$\frac{\mathbf{\Theta}(s)}{\mathbf{E}(s)} = \frac{K}{R(1 + sT_f)} \frac{1}{Bs[1 + T_m s]}$$

Required Transfer Function is

$$\frac{\mathbf{\Theta}(s)}{\mathbf{E}(s)} = \frac{K/(RB)}{s(1 + sT_f)(1 + T_m s)} \tag{9}$$

We shall denote this system transfer function as $\mathbf{H}(s)$.

- Impulse Response: Put an impulse as input into the system and calculate the output. Therefore set $e(t) = \delta(t) \Rightarrow \mathbf{E}(s) = 1$ Hence, to calculate output (which is the impulse response when an impulse is input)

$$\begin{aligned} \mathbf{\Theta}(s) &= \mathbf{H}(s)\mathbf{E}(s) = \mathbf{H}(s) \times 1 \\ &= \mathbf{H}(s) \end{aligned} \tag{10}$$

Therefore, the time domain impulse response $h(t)$ is just $\mathcal{L}^{-1}(\mathbf{H}(s))$
Remember: *System transfer function is Laplace transform of the impulse response, or the impulse response is the inverse Laplace Transform of the system transfer function.* So

$$h(t) = \mathcal{L}^{-1}\left\{\frac{K/(RB)}{s(1+sT_f)(1+T_m s)}\right\} \tag{11}$$

Need to use partial fractions to express the transfer function in a simpler form to make taking the inverse easier (i.e. to use tables).

$$\frac{K/(RB)}{s(1+sT_f)(1+T_m s)} = \frac{A}{s} + \frac{B}{(1+sT_f)} + \frac{C}{(1+sT_m)}$$

Using "Cover Up" rule

$$A = \frac{K/(RB)}{(1+0\times T_f)(1+T_m\times 0)} = \frac{K}{RB}$$

$$B = \left.\frac{K/(RB)}{s(1+T_m s)}\right|_{s=-\frac{1}{T_f}}$$

$$=$$

$$=$$

$$= \frac{-KT_f^2/(RB)}{(T_f - T_m)}$$

$$C = \qquad\qquad = \frac{K/(RB)}{-\frac{1}{T_m}(1-\frac{T_f}{T_m})}$$

$$= \frac{-KT_m/(RB)}{(1-\frac{T_f}{T_m}} = \frac{KT_m^2/(RB)}{(T_f - T_m)}$$

Hence (after some simplification)

$$\frac{K/(RB)}{s(1+sT_f)(1+T_m s)} = \frac{K}{RB}\left[\frac{1}{s} - \frac{T_f}{(T_f - T_m)}\left(\frac{1}{s+\frac{1}{T_f}}\right) + \frac{T_m}{T_f - T_m}\left(\frac{1}{s+\frac{1}{T_m}}\right)\right]$$

So now we can use tables to find the inverse Laplace Transform, hence

$$\mathbf{H}(s) = \frac{K}{RB}\left[\frac{1}{s} - \frac{T_f}{(T_f - T_m)}\left(\frac{1}{s+\frac{1}{T_f}}\right) + \frac{T_m}{T_f - T_m}\left(\frac{1}{s+\frac{1}{T_m}}\right)\right]$$

$$\Rightarrow \quad h(t) = \mathcal{L}^{-1}\left\{\mathbf{H}(s)\right\}$$

$$= \frac{K}{RB}\left[u(t) - \frac{T_f}{(T_f - T_m)} \qquad\qquad + \frac{T_m}{T_f - T_m} \qquad\qquad \right]$$

Consider:

- T_f generally smaller than T_m.
- Electronic cct transient FAST, Mechanical cct transient SLOW

- Step response $g(t)$ (say) : can get this either i) integrate the impulse response OR ii) set $e(t) = u(t)$ and then calculate output. You can spot that the only terms that involve t in $h(t)$ have simple exponentials, so integration seems straightforward

$$g(t) = \int_0^t h(\tau)d\tau$$
$$= \frac{K}{RB}\left[\int_0^t \left(u(\tau) - \frac{T_f}{(T_f - T_m)}e^{-\left(\frac{1}{T_f}\tau\right)} + \frac{T_m}{T_f - T_m}e^{-\left(\frac{1}{T_m}\tau\right)}\right)d\tau\right]$$

remember $\int e^{-t/a} = \frac{1}{-(1/a)}e^{-t/a} = -ae^{-t/a}$

$$= \frac{K}{RB}\left[\qquad\qquad\qquad\qquad \right]_0^t$$

$$= \frac{K}{RB}\left[t + \frac{T_f^2}{(T_f - T_m)}e^{-\left(\frac{t}{T_f}-1\right)} - \frac{T_m^2}{T_f - T_m}e^{-\left(\frac{t}{T_m}-1\right)}\right]$$

Steady state response is $g(t)$ as $t \to \infty$. So steady state response is

$$g_{t\to\infty}(t) = \qquad (12)$$