SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 3.

Dr Anil Kokaram
Electronic and Electrical Engineering Dept.
anil.kokaram@tcd.ie www.mee.tcd.ie\~ack

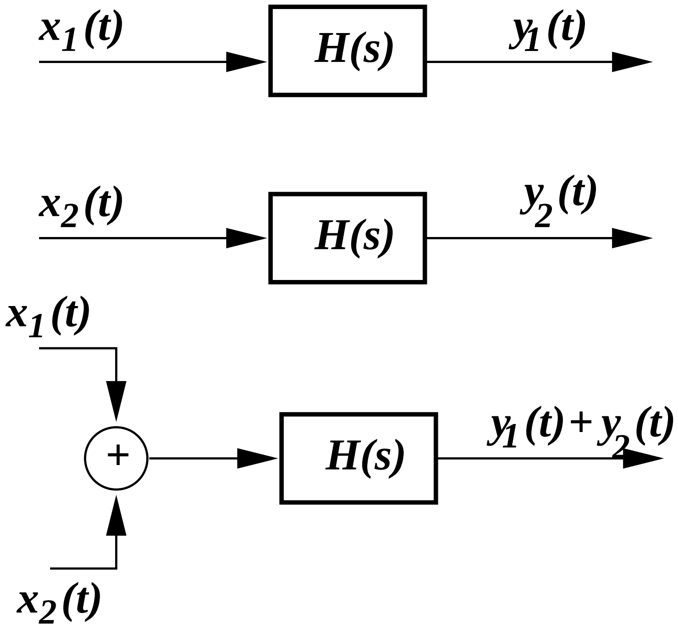
CONVOLUTION

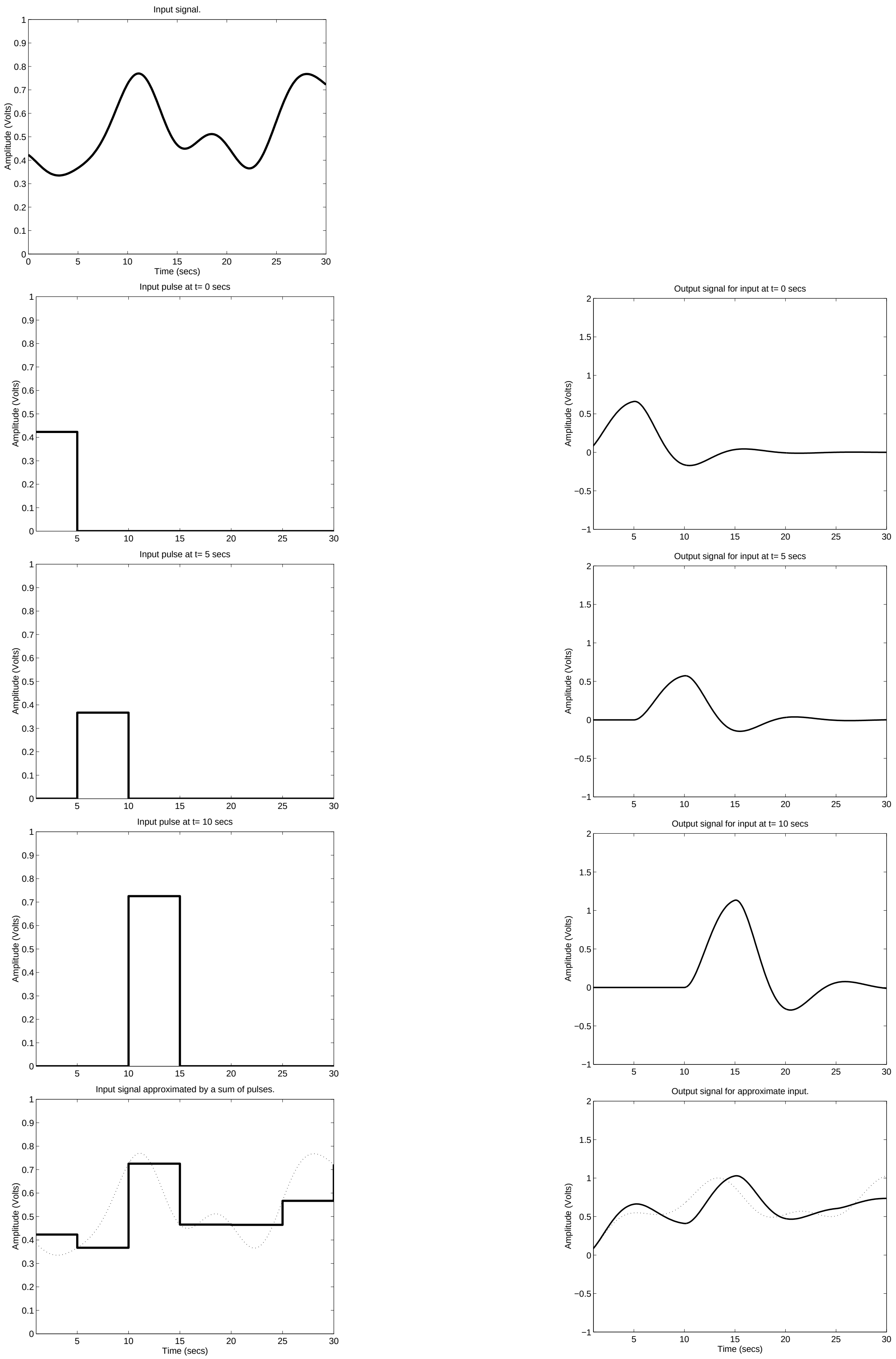
- Recall that at the start of the System Response discussion we said that there are situations when the Laplace Transform will not help.
- There is another way to look at the interaction of input signals with systems. When there are two ways to look at a problem it makes sense to explore both methods since they are likely to give useful and different insights to the solution of the problem.
- This alternative mechanism is a time domain (or spatial domain .. for images) approach to finding out what the response of a system is to an arbitrary input.
- The process is called *convolution*.
- It also is very useful for proving some basic truths about systems e.g. frequency response (later) and especially useful for quantifying exactly what happens when you digitize signals. (Like in CD and DVD players).

1 The basis of Convolution

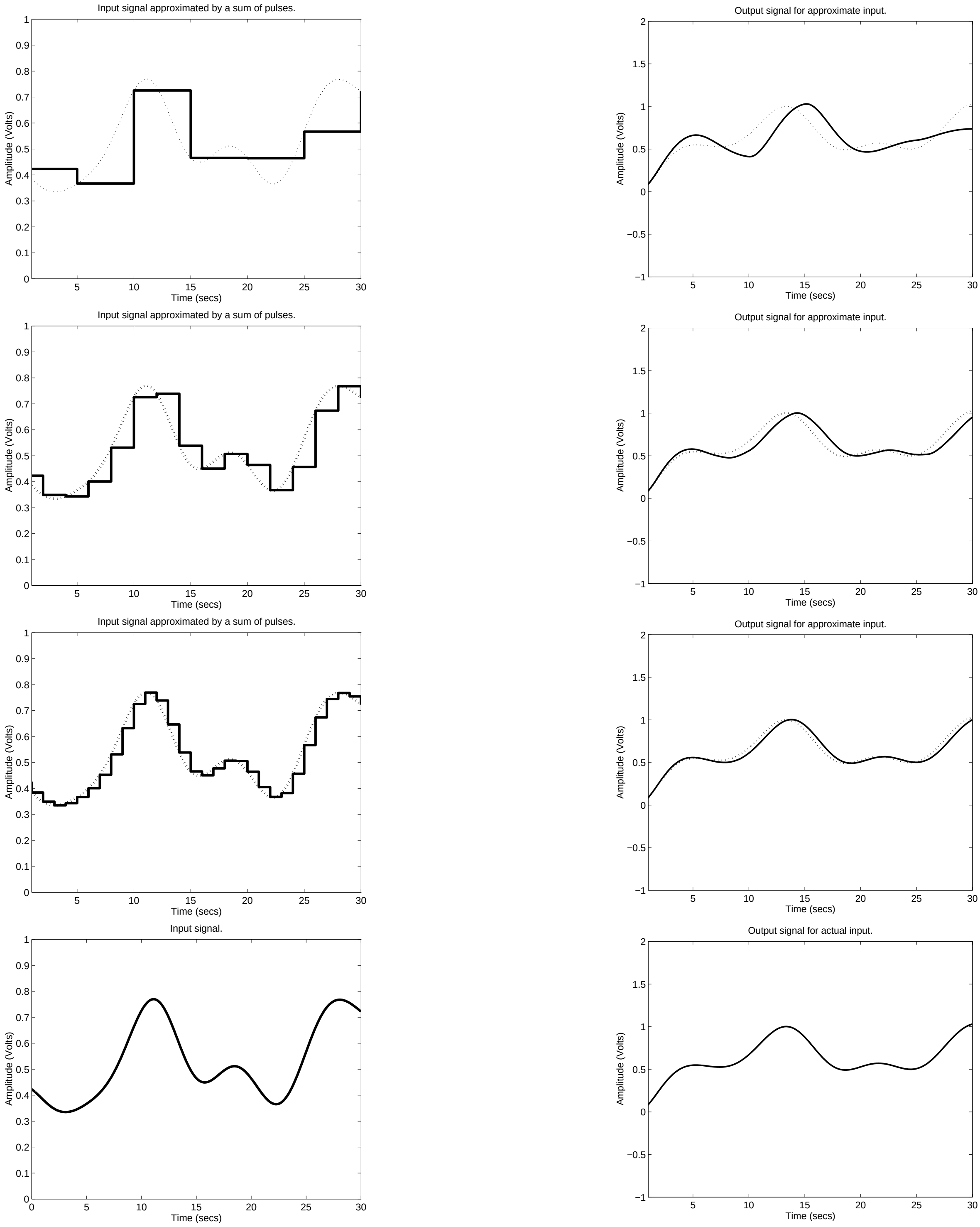
- Suppose we have found (by Laplace analysis if possible, or experiment, or someone tells us) the impulse response of a system $h(t)$.
- Can we use this simple information to work out the response of the system to **any** input?
- First of all : remember the tenets of LTI systems.

L stands for **Linear**. LTI systems obey property of superposition.
TI stands for **Time Invariant**. Parameters of system do not change with time. The response of the system to a given input remains the same no matter **when** that input occurs.




Input signal.
Amplitude (Volts)
Time (secs)
Input pulse at t= 0 secs
Output signal for input at t= 0 secs
Input pulse at t= 5 secs
Output signal for input at t= 5 secs
Input pulse at t= 10 secs
Output signal for input at t= 10 secs
Input signal approximated by a sum of pulses.
Output signal for approximate input.

Now take limit as $\triangle \rightarrow 0$



As $\Delta \rightarrow 0$, Pulses $\rightarrow$ Impulse functions (Delta Functions) and system response to pulses $\rightarrow$ system *impulse response*. The process of 'superposition' $\rightarrow$ CONVOLUTION.

2 The convolution integral

$$\textbf{INPUT} \longrightarrow \textbf{OUTPUT}$$

$$p_\Delta(t) \longrightarrow y_\Delta(t)$$

$$x(t)p_\Delta(t) \longrightarrow x(t)y_\Delta(t)$$

$$x(t)p_\Delta(t) + x(t-\Delta)p_\Delta(t-\Delta) \longrightarrow$$

$$\sum_{k=0}^{N} x(t-k\Delta)p_\Delta(t-k\Delta) \longrightarrow \sum_{k=0}^{N} x(t-k\Delta)y_\Delta(t-\Delta)$$

As $\Delta \to 0$, this means that $p_\Delta(t) \to \delta(t)$ and $y_\Delta(t) \to h(t)$ and $k\Delta \to \tau$, and output $= y(t)$.

$$x(t) \longrightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

HENCE THE CONVOLUTION INTEGRAL

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

1. NOTATION: $y(t) = h(t) * x(t)$
2. ALSO: $y(t) = \int x(t-\tau)h(\tau)d\tau$
3. Note that the arguments of $x(\cdot)$ and $h(\cdot)$ always add up to t !

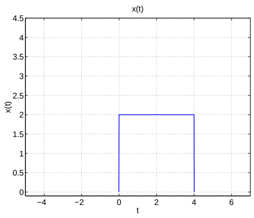
There are excellent graphical, interactive demonstrations of this at http://www.jhu.edu/∼signals/convolve/index.html

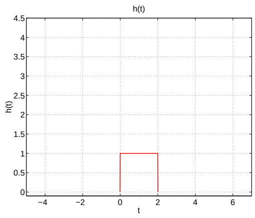
2.1 Simplifications for causality

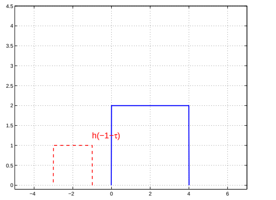
CAUSALITY simply requires that the system impulse response does not exist before $t = 0$. Thus $h(t) = 0$ for $t < 0$. In other words, the system **cannot** anticipate the arrival of an input signal and output a response *before* the signal arrives.

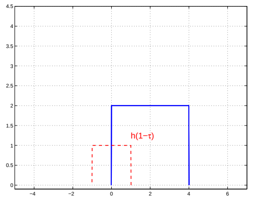
THUS

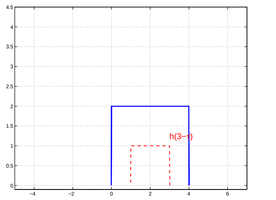
3 Example 1

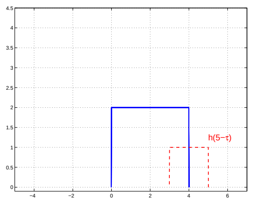


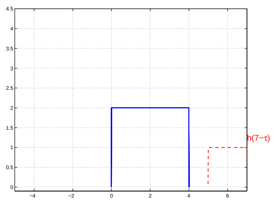


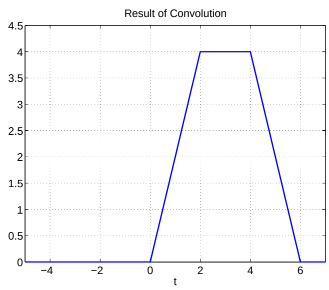












4 Example 1 Using Laplace Transforms

$$
\begin{aligned}
\mathbf{Y}(s) &= \mathbf{X}(s)\mathbf{H}(s) \\
&= \mathcal{L}\Big\{2(u(t) - u(t-4))\Big\}\mathcal{L}\Big\{u(t) - u(t-2)\Big\} \\
&= \left[\frac{2}{s} - \qquad\qquad \right]\left[\frac{1}{s} - \qquad\qquad \right] \\
&= 2\left[\frac{1}{s^2} - e^{-2s}\frac{1}{s^2} \qquad\qquad\qquad\qquad \right] \\
\Rightarrow y(t) &= \mathcal{L}^{-1}\Big\{\mathbf{Y}(s)\Big\} \\
&= 2\Big[r(t) - r(t-2) - \qquad\qquad\qquad\qquad \Big]
\end{aligned}
$$

Which looks like ...

5 Some points about Convolution

- Remember: Given a system transfer function, $\mathbf{F}(s)$, and a signal input $x(t)$; to get the output signal $y(t)$ do
  1. Calculate $\mathbf{X}(s)$
  2. Calculate the Laplace Xform of the output signal, $\mathbf{Y}(s) = \mathbf{X}(s)\mathbf{F}(s)$
  3. Then you can get the time domain output signal $y(t) = \mathcal{L}^{-1}\{\mathbf{Y}(s)\}$
- Remember: The Laplace transform of a system's impulse respose is the system transfer function. So if $x(t) = \delta(t)$ is presented as system input, then the corresponding system output is the impulse reponse $h(t)$.
- Remember: From the impulse response $h(t)$ we can get the system transfer function $\mathbf{F}(s) = \mathbf{H}(s)$.
- Given a system impulse response, now have several ways of calculating the output signal $y(t)$ when a system is presented with an arbitrary input $x(t)$. Convolution, or use the Laplace Xform.
- Usually easier to work with $\mathbf{H}(s)$ rather than $h(t)$. (MULTIPLICATION EASIER THAN INTEGRATION).
- Convolution easy if $x(t)$ or $h(t)$ consists of impulses. (Happens in signal processing and communications, will introduce this later.)
- Convolution useful for proving some general results e.g. frequency response.
- In a sense convolution is the principle used in the application of *digital* filters.

The system impulse reponse is all you need to know to completely characterise the system bahaviour given any input. From the impulse response, you can get the step response, the system transfer function (in theory), and using convolution, you can directly calculate the response to any arbitrary input.

6 Transform of a Convolution

$$\mathcal{L}\Big(h(t) * x(t)\Big) = \mathcal{L}\Big(h(t)\Big) \times \mathcal{L}\Big(x(t)\Big) \quad (1)$$

PROOF:

$$\begin{aligned}\mathcal{L}\Big(x(t) * h(t)\Big) &= \mathcal{L}\Big(\int_{-\infty}^{\infty} h(t-u)x(u)du\Big) \\ &= \int_{0}^{\infty} e^{-st}\Big(\int_{-\infty}^{\infty} h(t-u)x(u)du\Big)\, dt\end{aligned}$$

Changing the variable of integration :

$$\begin{aligned} &= \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s(u+v)}h(v)x(u)du\ dv \\ &= \\ &= \mathbf{H}(s) \times \mathbf{X}(s) \quad (2)\end{aligned}$$

(remember we make the assumption that the system is causal (so $h(v) = 0$ for $v < 0$) and the signal is 'causal' i.e. $x(u) = 0$ for $u < 0$.)

In general,

THE LAPLACE TRANSFORM OF THE CONVOLUTION OF TWO SIGNALS $x(t)$ and $y(t)$ IS THE PRODUCT OF THE LAPLACE TRANSFORMS OF THE SIGNALS. Thus

$$\mathcal{L}\Big(x(t) * y(t)\Big) = \mathbf{X}(s) \times \mathbf{Y}(s) \quad (3)$$