SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 4.
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STABILITY

- Have looked at modeling dynamic systems using differential equations. and used the Laplace transform to help find step and impulse responses
- Use of the Laplace Transform may be summarized through Transfer functions. The contents of the blocks in the block diagrams are Transfer functions.
- Looked at alternative system analysis in using convolution. Found that it may sometimes be more complicated than Laplace analysis, however does yield useful insights to system behaviour.
- Now we need to assess the stability of systems. In general, only stable systems are useful to us.
- System stability can be assessed in both s-plane and in the time domain (using the system impulse response).
- This handout will
  1. Define *asymptotic* and *Bounded-input, Bounded-output (BIBO)* stability. Then examine equivalence for Linear Systems.
  2. Relate system stability to poles of transfer function.
  3. Relate transient response to poles of transfer function.

1 The need for stability

Some questions may occur to you:
Who cares if a system is unstable? Is being unstable bad? Why is being stable good? What's the big deal? What does it mean for a system to be stable anyway?

Consider a Hovering Harrier.

- The pilot switches on *hover* mode. He expects the control system to adjust the lift surfaces and the directional thrust jets so that the aircraft remains in the same place. Why is this so difficult?
- Its because without the engine thrust, the vehicle will fall. Because of wind gusts you have to compensate for the applied force to keep the aircraft in the same place.
- Too much thrust and you push the plane off the hover state, too little and the same thing happens. Thrust in the wrong direction causes more problems.
- So what's the biggie? You just have to *get it right*.
- That's the bit about stability ... in this case the ability to maintain a steady state regardless of external *noises*.

Consider a DVD Player, or a CD Player.

- Data is read off the disk using optical technology.
- The combination of a controlled speed and correct targeting/focusing of the beam, yields a regular flow of data.
- The steady speed is only possible by overcoming the inertia of the disk material and the friction of the bearings.
- Too much speed and you don't have enough time to read the data from each spot. Too little and your operating system times out because its taking too long to get to the next bit.
- Stability here is about maintaining a steady speed no matter what else is going on e.g. somebody kicks the case, or you turn the player on end, or you run around with the CD player and shake the bearings.

It'll be pretty sad if testing a system for stability implies you have to test it with all possible inputs to see if it performs as you expect or not. Thankfully, the unifying idea in the LTI systems above is that when a stable system is in some steady state and you *kick it* (within some reasonable bounds), it returns to a steady state after a little while. The response of a system to a kick (an impulse) is its impulse response. Stabilty can therefore be determined by a system impulse response.

Asymptotic Stability

Definition:
A linear system is *asymptotically stable* if its impulse response $h(t)$ satisfies the condition

$$\int_0^\infty |h(t)|dt = B < \infty \tag{1}$$

where B is some positive number.

Examples:

1. LCR circuit: $h(t) = e^{-2t}\sin(3t+4)$
2. Delay line with lossy reflections:

$$h(t) = \sum_{k=0}^{\infty} \frac{1}{2^k}\delta(t-kT)$$

2 Bounded-Input Bounded-Output (BIBO) Stability

Definition:
A linear system is *BIBO stable* if there is a positive number B such that, for any bounded input signal $x(t)$, $|x(t)| < X$, the resulting output signal $y(t)$ is bounded by: $|y(t)| < XB$.

Theorem:
If a linear system is asymptotically table, then it is also BIBO stable.

Proof:

$$\begin{aligned}
|y(t)| &= \left| \int_0^t h(\tau)x(t-\tau)d\tau \right| \\
&\leq \int_0^t |h(\tau)||x(t-\tau)|d\tau \\
&< X \int_0^t |h(\tau)|d\tau \\
&\leq XB
\end{aligned}$$

Theorem:
If a linear system is BIBO stable then it is also asymptotically stable.

Proof:
The idea is to find an input that excites the system as much as possible, but is itself bounded. Consider an interval $0 \leq t \leq T$ and choose

$$x(T - t) = \text{sign}[h(t)]$$

Then, by the convolution integral

$$|y(T)| = \int_0^T |h(t)| dt$$

But the input we have chosen satisfies $|x(t)| = 1$, so if the system is BIBO stable then necessarily

$$\int_0^\infty |h(t)| dt < B$$

for some B.

So: *Asymptotic and BIBO stability are equivalent for linear systems.*

Note: They are not equivalent for non-linear systems.

3 Marginal Stability

Definition:

A linear system is *marginally stable* if it is not asymptotically stable and one can find $A, B < \infty$ such that

$$\int_0^T |h(t)|dt < A + BT \text{ for all } T$$

3.1 Examples:

1. Integrator

$$h(t) = u(t)$$

2. Delay line with *lossless* reflections

$$h(t) = \sum_{k=0}^{\infty} \delta(t-k)$$

3. Undamped second order system

$$h(t) = \cos(3t)$$

4 Instability

Definition:

A system is *unstable* if it is neither asymptotically stable nor marginally stable.

4.1 Examples:

1. Inverted Pendulum:

$$h(t) = e^{4t} + e^{-4t}$$

2. Two integrators in series:

$$h(t) = t$$

3. Unstable oscillator:

$$h(t) = e^{0.01t}\sin(0.3t)$$

Warning[1]: Different people use different definitions of stability. In particular, systems which we have defined to be marginally stable would be regarded as stable by some and unstable by others. For this reason we avoid the term 'stable' without qualification.

[1] Will Robinson

Poles and Zeros

The *zeros* of a transfer function $\mathbf{G}(s)$ are those values of s at which $\mathbf{G}(s)$ becomes zero, and its *poles* are those values of s at which $\mathbf{G}(s)$ becomes infinite.

If a transfer function is *rational*, i.e. it can be written as the ratio of two polynomials

$$\mathbf{G}(s) = \frac{n(s)}{d(s)}$$

then the zeros of $\mathbf{G}(s)$ are the roots of the numerator polynomial $n(s)$, and its poles are the roots of the denominator polynomial $d(s)$. $d(s)$ is also known as the *characteristic polynomial*.

For physically realisable systems:

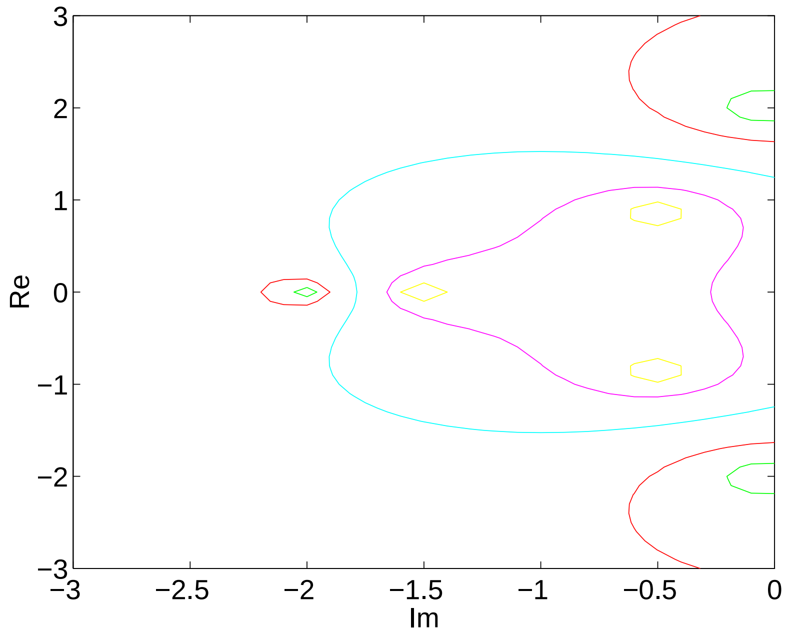
$$\deg[n(s)] \leq \deg[d(s)]$$

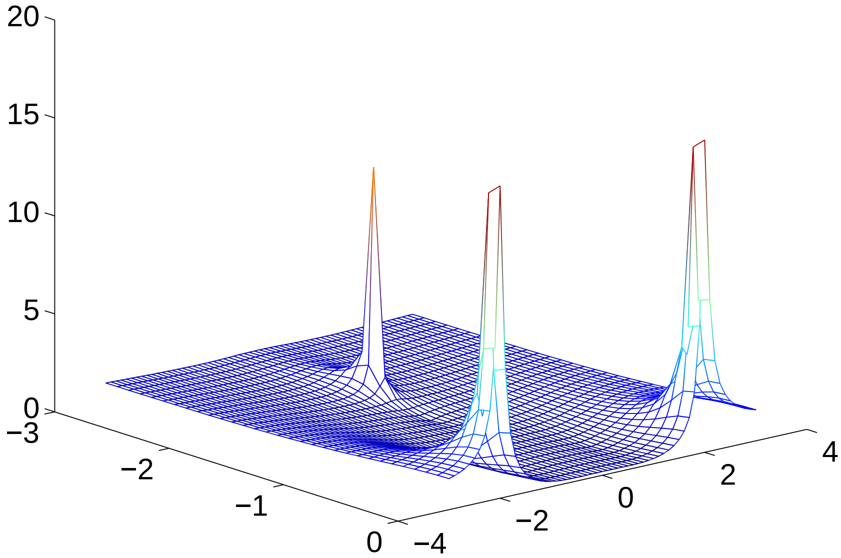
Any system whose transfer function violates this condition is not asymptotically stable.

The poles of a transfer function are also called the *characteristic roots* or *auxiliary roots*.

We can also speak of the poles and zeros of the Laplace transform of a *signal*.

s-plane surface for $\mathbf{G}(s) = \frac{(s+1.5)(s^2+s+1)}{(s+2)(s^2+0.1s+4)}$





5 Poles and Stability

If a system $\mathbf{H}(s)$ is asymptotically stable then, for $\text{Re}(s) \geq 0$,

$$\begin{aligned}
|\mathbf{H}(s)| &= \left| \int_0^\infty e^{-st} h(t) dt \right| \text{ (Laplace transform of impulse response } h(t) \text{ is transfer function } \bar{h}(s)) \\
&\leq \int_0^\infty |e^{-st}||h(t)| dt \\
&\leq \int_0^\infty |h(t)| dt \text{ (since } |e^{-st}| \leq 1 \text{ for } \text{Re}(s) \geq 0 \\
&\leq B \text{ Because system is asymptotically stable} \\
&< \infty
\end{aligned}$$

Hence:

The transfer function of an asymptotically stable system cannot have any poles in the right half of the complex plane or on the imaginary axis.

5.1 Alternative Derivation

Assume a rational transfer function:

$$\mathbf{H}(s) = r_0 + \sum_{i=1}^{N} \frac{r_i}{(s-p)^{v_i}}$$

(Usually the poles (i.e. the values p_i) are distinct so that $v_i = 1$).

Hence the impulse response:

$$h(t) =$$

Now consider a single term of this sum:

$$h_i(t) = r_i \frac{t^{(v_i-1)}}{(v_i-1)!} e^{p_i t}$$

Let $p_i = \sigma_i + j\omega_i$. Then

$$|h_i(t)| = \frac{|r_i|}{(v_i-1)!} t^{v_i-1} e^{\sigma_i t}$$

If $\sigma_i < 0$ then $|h_i(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Also if $\sigma_i < 0$:

$$
\begin{aligned}
\int_0^\infty |h_i(t)|dt &= \int_0^\infty |h_i(t)|e^{-st}dt \Big|_{s=0} \\
&= \mathcal{L}\Big\{|h_i(t)|\Big\}\Big|_{s=0} \\
&= \frac{|r_i|}{(s-\sigma_i)^{v_i}}\Big|_{s=0} \\
&= \frac{|r_i|}{(-\sigma_i)^{v_i}} \\
&< \infty
\end{aligned}
$$

If $\sigma_i = 0$ and $v_i = 1$:

$$|h_i(t)| = |r_i| \text{ (a constant)}$$

Hence:

$$\int_0^\infty |h_i(t)|dt = \infty$$

but:

$$\int_0^T |h_i(t)|dt \leq |r_i|T$$

So the system exhibits **marginal stability**.

If $\sigma_i = 0$ and $v_i > 1$:

$$\int_0^T |h_i(t)|dt = \frac{|r_i|T^{v_i}}{v_i!}$$

and the system is **unstable**.

If $\sigma_i > 0$ then the system is also **unstable.**

Stability Theorem

1. A system is asymptotically stable if *all* its poles have negative real parts.
2. A system is unstable if *any* pole has a positive real part, *or* if there are repeated poles on the imaginary axis.
3. A system is marginally stable if all the poles on the imaginary axis are distinct, *and* all the remaining poles have negative real parts.

6 Poles and transient responses

Impulse response:

$$h(t) = r_0\delta(t) + \sum_{i=1}^{N} r_i \frac{t^{(v_i-1)}}{(v_i-1)!}e^{p_i t}$$

Consider $e^{p_i t}$.

p_i real: real exponential with time constant $1/p_i$.

p_i complex: always has complex conjugate pole p_i^*.

These combine to give a damped or growing sinusoid.

Assume $v_i = 1$, then for a second order system $(i = 2)$:

$$\mathcal{L}^{-1}\left\{\frac{Ae^{j\phi}}{s-p_i} + \frac{Ae^{-j\phi}}{s-p_i^*}\right\} = Ae^{j\phi}e^{p_i t} + Ae^{-j\phi}e^{p_i^* t}$$

Recall that we defined $p_i = \sigma_i + j\omega_i$

$$= Ae^{\sigma_i t}\left\{e^{j(\omega_i t+\phi)} + e^{-j(\omega_i t+\phi)}\right\}$$

$$= 2Ae^{\sigma_i t}\cos(\omega_i t + \phi)$$

From the standard form of the characteristic polynomial (for a second order system):

$$(s - p_i)(s - p_i^*) \equiv s^2 + 2c\omega_n s + \omega_n^2$$

(c: damping factor, ω_n: undamped natural frequency)
we see:

$$-2\mathrm{Re}\{p_i\} = 2c\omega_n \text{ and } |p_i|^2 = \omega_n^2$$

Hence:

$$c = \frac{-\mathrm{Re}\{p_i\}}{|p_i|} \text{ and } \omega_n = |p_i|$$

So from the positions of the poles in the s-plane we can deduce some of the characteristics of the 2nd order system.

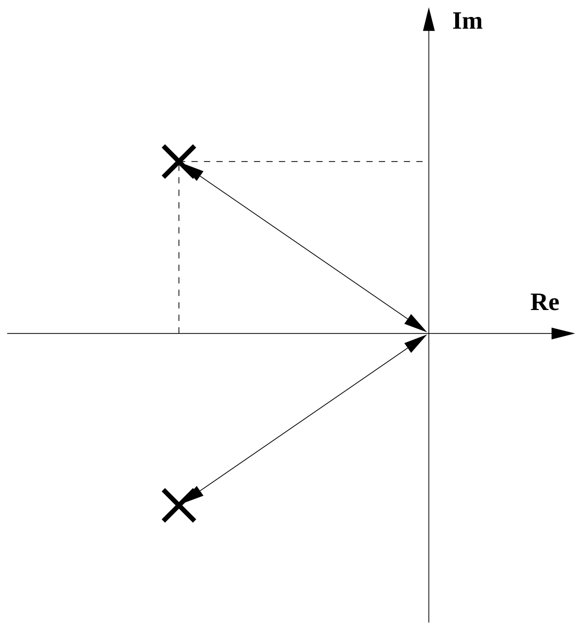


Figure 1: Second order system showing relationship of pole locations to c, ω_n

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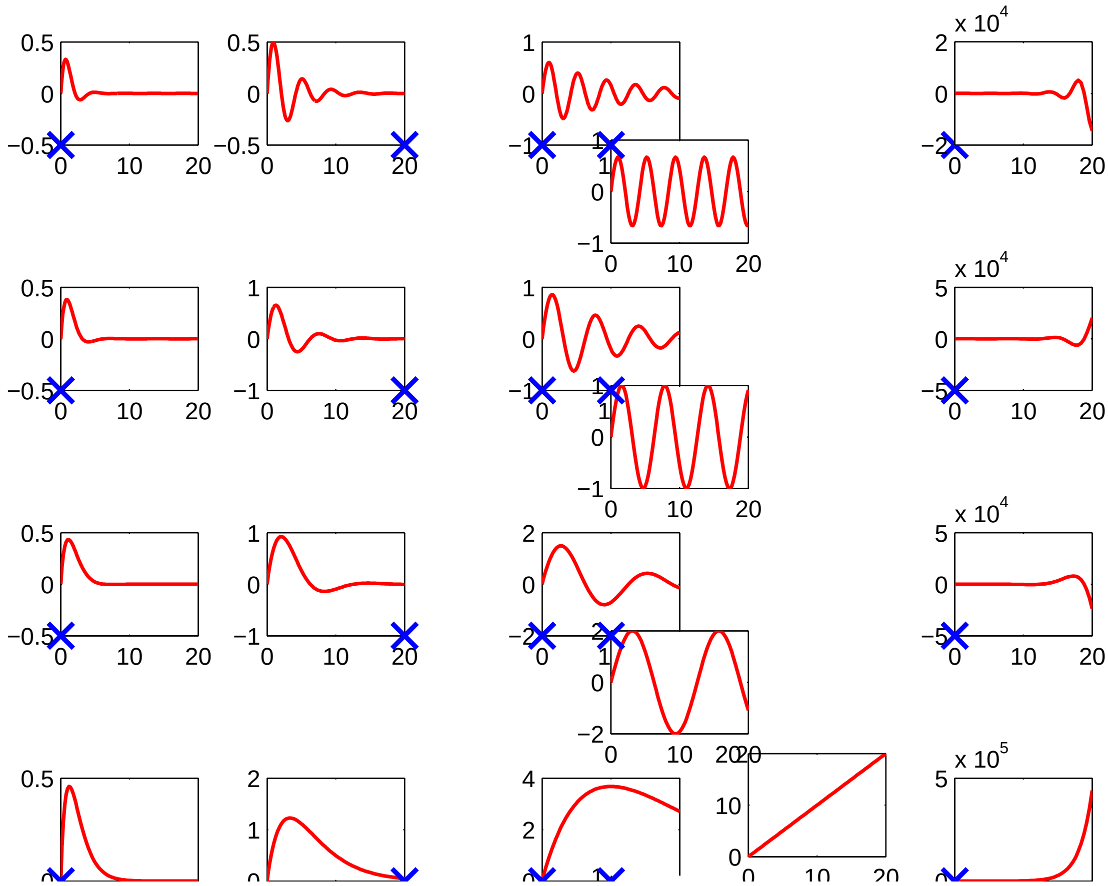
The system pole locations are related to various impulse responses below. **This is an important figure**.

Note: The real part of the pole, σ_i determines stability and the time constant, $|1/\sigma_i|$.

The imaginary part ω_i determines the the natural frequency ω_i (rad/sec).



You can interactively explore the effect of pole and zero positions on impulse responses at http://www.jhu.edu/∼signals/explore/index.html

7 A last word on time domain behaviour: THE FINAL VALUE THEOREM

Transient system behaviour is only part of the story as far as time domain system behviour is concerned. Of equal interest is finding out the final value of a signal when the system has settled down to steady state behaviour. The final value theorem is a simple mechanism for using the Laplace Transform of a signal to predict its final value as $t \to \infty$.

Given some signal $f(t)$, the final value theorem relates the steady state behaviour $f(t)$ to the behaviour of $\mathbf{F}(s)$ in the neighbourhood of $s = 0$. It states that

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \tag{2}$$

The conditions that need to be obeyed for this theorem to be successfully applied are as follows.

1. $\lim_{t \to \infty} f(t)$ exists. Which just means that $f(t)$ does indeed converge to some definite value as $t \to \infty$.
2. All poles of $s\mathbf{F}(s)$ are in the left half plane. Note that we are talking here about $\mathbf{s} \times \mathbf{F}(\mathbf{s})$ *not just* $\mathbf{F}(s)$.
3. $s\mathbf{F}(s)$ has no poles on the imaginary axis. Note that we are talking here about $\mathbf{s} \times \mathbf{F}(\mathbf{s})$ *not just* $\mathbf{F}(s)$.

Proof

Recall that (from tables for instance)

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s\mathbf{F}(s) - f(0)$$

We want to examine what happens when this first differential tends to zero, since then that would mean that the signal is no longer varying with time. Consider

$$\begin{aligned}
\lim_{s\to 0} s\mathbf{F}(s) - f(0) &= \lim_{s\to 0}\int_0^\infty \left(\frac{d}{dt}f(t)\right)e^{-st}dt \\
&= \int_0^\infty \left(\frac{d}{dt}f(t)\right)\lim_{s\to 0} e^{-st}dt \\
&= \int_0^\infty \left(\frac{d}{dt}f(t)\right)dt \\
&= \Big[f(t)\Big]_0^\infty \\
&= \lim_{t\to\infty} f(t) - f(0)
\end{aligned}$$

We can cancel $f(0)$ as it appears on both sides and we are left with

$$\lim_{s\to 0} s\mathbf{F}(s) = \lim_{t\to\infty} f(t)$$

7.1 How is this theorem actually used?

Well, its like this. The theorem relates the steady state value of any signal to the behaviour of $s\mathbf{F}(s)$ near $s = 0$. So if you are asked for instance, what is the steady state value of the step response of some system, you first have to calculate the Laplace transform of that step response. This would mean calculating the product of the system transfer function with $1/s$, or integrating the time domain impulse response and take the Laplace Transform. Then you can apply the theorem to work out what the final value of the step response is.

Alternatively, you can forget about the theorem and hope that you can always spot the final value of a time domain signal by manipulating it so that you can work out $\lim_{t\to\infty} f(t)$. For the step response example, that would mean working out the time domain signal output by taking the inverse laplace transform. Then trying to re-write the signal expression to handle $t \to \infty$ for all the terms in t. Sadly, it is not always easy to massage the final value out of a time domain expression. Hence you tend to have to know the final value theorem.