SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 5.
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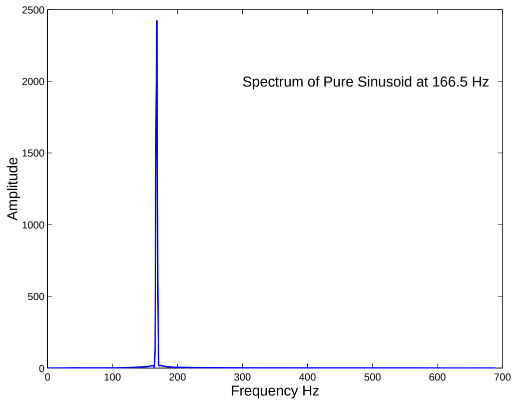
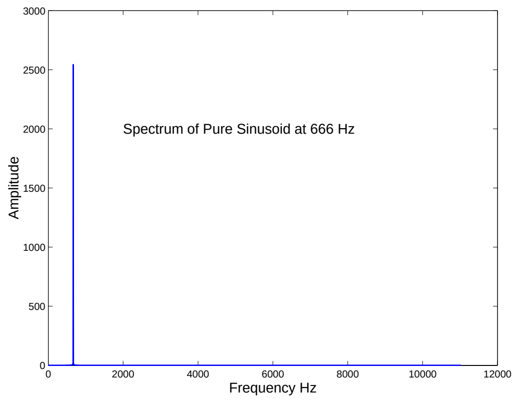
FREQUENCY RESPONSE

The first part of this handout gives some anecdotal evidence for a number of ideas that we will deal with in more mathematical detail in this and the next handout. The second part of this handout deals with system frequency response.

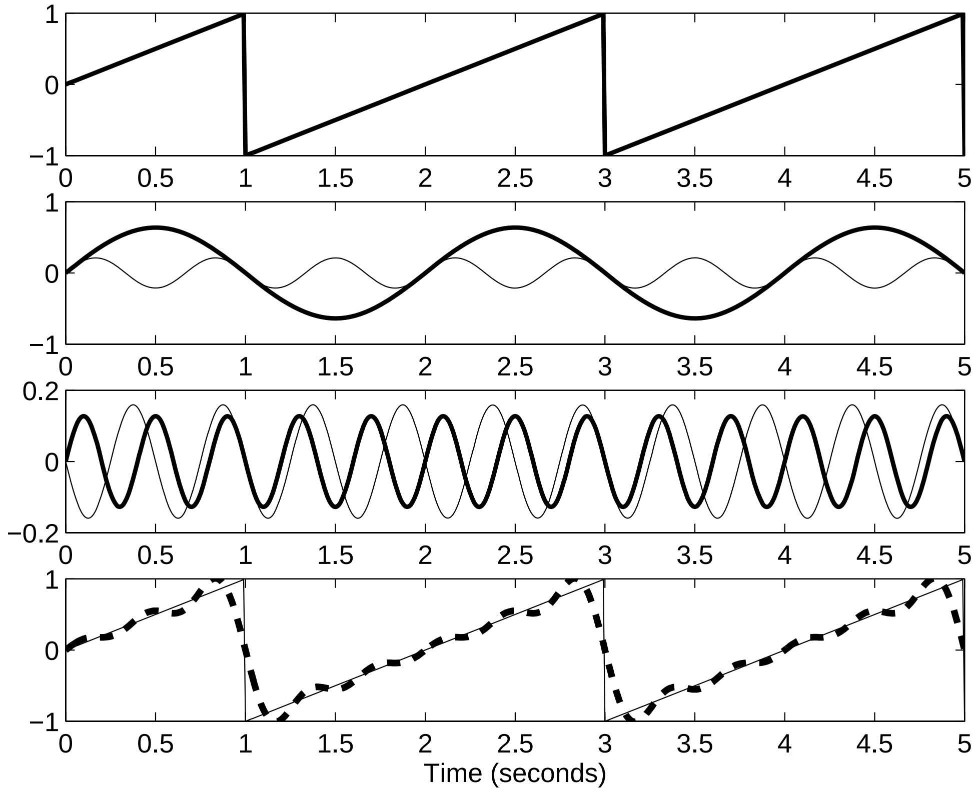
- Most people intuitively appreciate that 'sounds' consist of several 'frequencies' combined together. In fact, we hear and perceive sounds in terms of **both** frequency and time behaviour. Considering signals in terms of frequency content is another aspect of any system design.
- Consider some pure sinusoids at $f_1 = 666$ Hz and $f_2 = 166.5$ Hz, $x_1 = \sin(2\pi f_1 t)$, $x_2 = \sin(2\pi f_2 t)$. In this lecture, you will hear both of these tones played. Intuitively, if we wanted to express these signals in terms of frequency, we would like to be able to show plots as below.





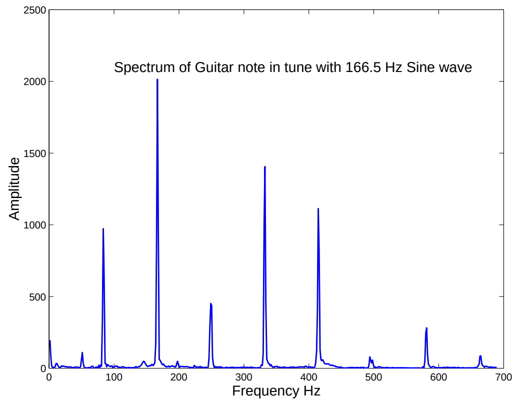
- These are plots of the signal *spectra*, $X_1(f)$, $X_2(f)$. Spectral information can be summarized graphically by the frequency spectrum of $x(t)$ (called $X(\omega)$ or $X(f)$).
- But what about a tone from a guitar? In the lecture you will hear two guitar tones that are in *tune* with the pure sinusoids above. But we know that the guitar sounds different from the pure sinusoid. Intuitively, you say that the guitar somehow as *more frequencies* in it than the pure sinusoid. What does that mean really?

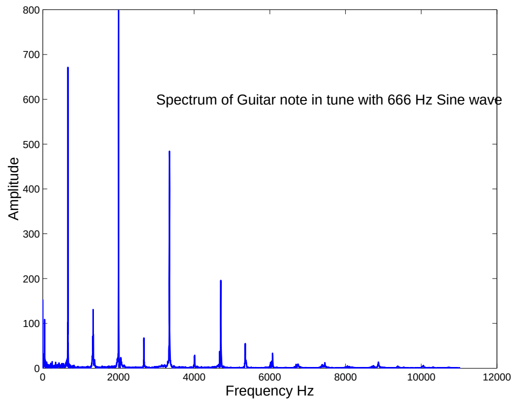
- It is in fact true to say that **all** *signals* can be expressed as a sum of sinusoidal signals of different frequencies.



- This amazing result is created through *Fourier Analysis*, which allows us to calculate the spectrum of any signal. We will examine this in detail in the next handout i.e. how exactly one can *quantify* the frequency content of a signal.

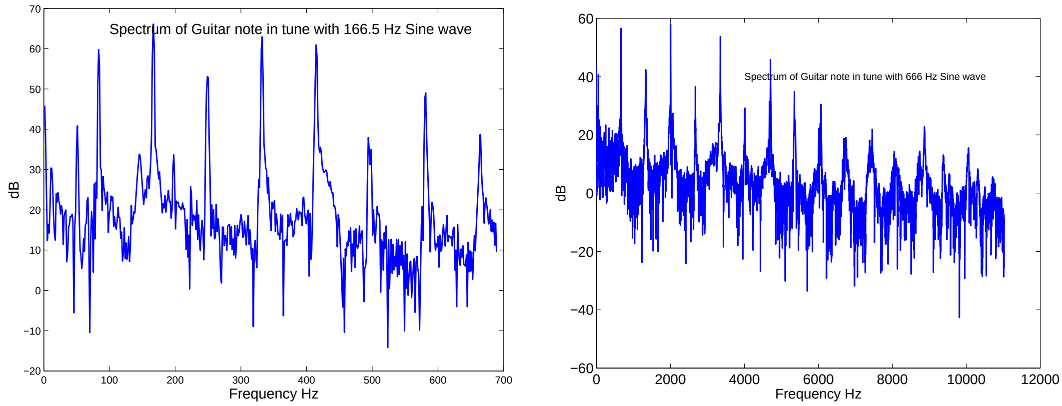
- Returning to the guitar tone, we can calculate their frequency spectra as below.





- You can see some interesting things
  - There is more than one frequency present in the signal
  - These frequency components seem to be equally spaced in frequency
  - The components do not all have the same amplitude
  - The first component is the one corresponding to the pure sinusoid that we heard initially.
- Some questions to answer: what's the difference between the high guitar note and the low one? What's the same? There is alot of background noise (hiss) in the guitar recording, where is that in the spectra?

- About Decibels. We will be using dB for spectra and spectral response because in many cases its the low amplitude behaviour that is very important for perception and performance. A dB scale emphasises small numbers and compacts large ones. Remember $dB = 20 \log 10(amplitude)$ See the guitar spectra below. Note how the dB scale shows spectral components that would be hard to see otherwise.



- A spectrogram shows the evolution of frequency content with time. It can be calculated by taking overlapping slices of the signal in time and calculating the spectra for each slice.

  PLEASE DO NOT IGNORE THE S1 AND S2 LABS, THEY ACTUALLY LEAD YOU THROUGH MANY OF THESE IDEAS AND CAN BE USED AS REVISION TOOLS FOR THE EXAMINATION

1 A problem

To help you get your mind around why we are doing what we are doing: let's set up a problem that contains many signal processing issues. Suppose we have a voice recording corrupted with noise (hiss) and a hum from some interfering equipment. The problem is to process this signal to remove the corruption of hum and if possible the noise as well. Think about the problem in terms of the frequency domain.

We can model the system causing the corruption as follows

$$y(t) = x(t) + f(t) + e(t)$$

where $y(t)$ is the observed, degraded signal; $x(t)$ is the original, clean audio signal, $f(t)$ is the corrupting hum, and $e(t)$ is random hiss component. We shall ignore the hiss in this part of the course, analysing that requires an understanding of statistical signal processing. So our degradation model is

$$y(t) = x(t) + f(t)$$

The question is, can we somehow remove the signal $f(t)$ from $y(t)$ to leave $x(t)$? Think about all you know about LTI systems so far ...

2 Frequency Response

- We know that we can *hear* the hum $f(t)$, so perhaps its sensible to think that we can remove it by thinking of the signal in terms of frequency content.
- We have seen that for LTI systems if one scales the amplitude of an input signal to a system then the output signal scales by the same amount. Hmm .. that's not going to help us so much maybe.
- Most of us already have an intuitive notion that systems respond differently to input signals which have different *frequencies* even if the amplitude of the signal applied to the system is the same, i.e. many systems

have a resonant frequency of input at which their response is strongest, while for other input frequencies their response is poor.

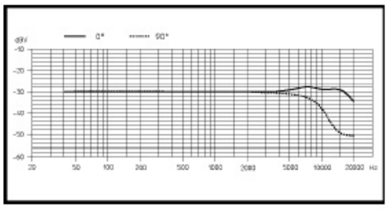
- The most common example of the manipulation of the frequency content of a signal in electrical systems is the home stereo system. The loudspeaker is designed to make sure that the sound is not distorted, and the 'graphic equalizer' is manipulated manually to emphasize aspects of the sound pleasing to the listener. One can boost the 'base tones' (low frequencies) or the 'treble tones' (high frequencies). 'Sharpness' and 'contrast' adjustment on your TV set is a manifestation of the same principle applied to images.

- What we can seek to do to solve our problem is to design a system such that it responds well to the speech part, but somehow suppresses the hum signal. So we need to know more about frequency response of systems and analysis of frequency content of signals. You've been doing stuff with systems in the handouts so far, so lets look at system frequency response first.

- This handout explores quantitatively how *systems* **respond** to *signals* of different frequencies.

Microphone response   sennheiser.co.uk

ME 62   SENNHEISER



ME 62 Omni-Directional Microphone Head

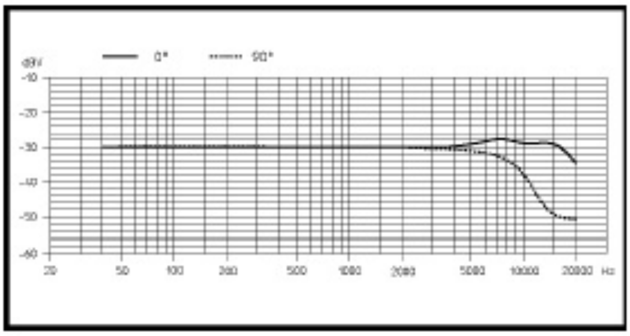
The ME 62 is an omni-directional microphone head suitable for K6 and K6P powering modules. It can be used for reporting, discussions and interviews. The ME 62 is particularly suitable for good reproduction of 'room' ambience and 'spaced omni' stereo recording. Matt black, anodised, scratch-resistant finish.

Features—Benefits

- Omni-directional pick-up pattern
- Minimal inherent self-noise
- Excellent rejection of rumble, wind and handling noise
- Wide frequency response
- High maximum sound pressure level
- Integral pop filter

Technical Data

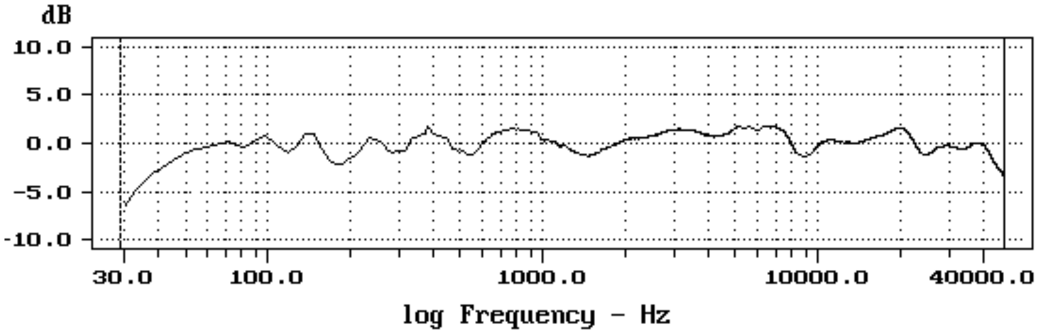
Pick-up pattern	omni-directional
Frequency response	20–20,000 Hz ± 2.5 dB
Sensitivity (free field, no load) (1 kHz)	31 mV/Pa ± 2.5 dB
Nominal impedance	200 Ω (with K6)
Min. terminating impedance	1000 Ω (with K6)
Equivalent noise level	
A-weighted (DIN IEC 651)	15 dB
CCIR-weighted (CCIR 468-3)	27 dB

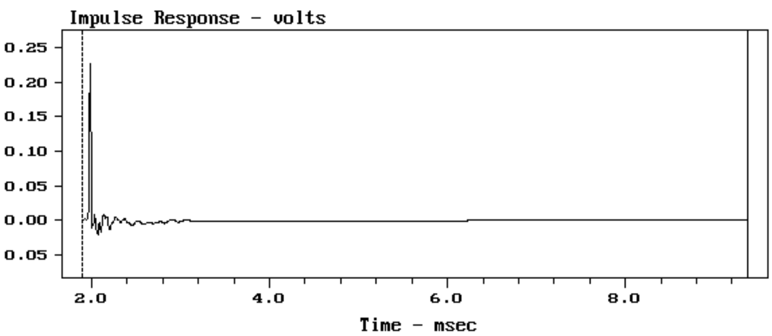


Loudspeaker earthworks.com

Click an image to see full size

Specifications:	
Frequency Response	40Hz to 40kHz ±2dB
Impedance	8 Ohm
Sensitivity	87 dB 1W/1m
Power Handling	150 watts continuous, 400 watts peak
Weight	32 pounds
Shielding	Fully Magnetically Shielded





2.1 Previsouly

- Already seen this in previous courses about electrical circuit behaviour e.g. AC cct analysis, reactance $1/j\omega C, j\omega L$
- Basic idea is that if a sinusoidal signal is input to a linear, asymptotically stable system; then the output will be also sinusoidal, WITH THE SAME FREQUENCY, but having a different magnitude and phase.

FREQUENCY RESPONSE CAN BE **MEASURED**. Just apply every possible signal input frequency and then measure the output frequency and 'phase'.

You actually investigate this in Laboratory S1. Please do not ignore this laboratory, it is meant to complement the coursework.

2.2 Frequency Response : Quantitative Analysis

- Want to predict the frequency response of systems analytically. Eventually you will then be able to *design* a system to have a desired frequency response.
- Basic Result is as follows:

$$\frac{Y}{X} = \left|\mathbf{H}(j\omega)\right|$$
$$\phi = \arg\{\mathbf{H}(j\omega)\}$$

- So we can COMPUTE frequency response $H(\omega)$ from the transfer function $\mathbf{H}(s)$ by setting $s = j\omega$ in the transfer function.
- ALSO we can estimate the transfer function by measuring the frequency response.

2.3 Meaning of steady state

TWO USES OF THE TERM STEADY STATE

1. Steady state response to given sinusoidal input.
2. Steady state gain i.e. D.C. Gain (given by $\mathbf{H}(0)$).
3. BOTH MEAN **WHEN TRANSIENTS HAVE DIED DOWN.**

2.4 Proof of Basic Result

Suppose we are given a system with impulse response $h(t)$, and we present this system with an input signal $x(t) = \cos(\omega t)$. We want to work out what the output signal $y(t)$ is going to be. (Going to have to remember your complex number algebra again ...)

Using convolution

$$
\begin{aligned}
y(t) &= \int_0^t x(t-\tau)h(\tau)d\tau \\
&= \int_0^t \cos(\omega(t-\tau))h(\tau)d\tau
\end{aligned}
$$

Now $\cos(\omega(t-\tau) = \mathrm{Re}[e^{j\omega(t-\tau)}]$, so

$$
\begin{aligned}
&= \mathrm{Re}\left\{\int_0^t e^{j\omega(t-\tau)}h(\tau)d\tau\right\} \\
&= \mathrm{Re}\left\{\int_0^\infty e^{j\omega t}e^{-j\omega\tau}h(\tau)d\tau - \int_t^\infty e^{j\omega t}e^{-j\omega\tau}g(\tau)d\tau\right\} \\
&= \mathrm{Re}\left\{e^{j\omega t}\left[\int_0^\infty e^{-j\omega\tau}h(\tau)d\tau - \int_t^\infty e^{-j\omega\tau}g(\tau)d\tau\right]\right\}
\end{aligned}
$$

Remember $\mathbf{H}(s) = \int_0^\infty h(\mu)e^{-s\mu}d\mu$, so

$$
= \mathrm{Re}\left\{e^{j\omega t}\mathbf{H}(j\omega)\right\} - \underbrace{\mathrm{Re}\left\{e^{j\omega t}\int_t^\infty e^{-j\omega\tau}h(\tau)d\tau\right\}}
$$

Continuing ...

Given

$$x(t) = \cos(\omega t) = \text{Re}\left\{e^{j\omega t}\right\}$$

We have shown that, after transients have decayed,

$$\Rightarrow y(t) = \text{Re}\left\{e^{j\omega t}\mathbf{H}(j\omega)\right\}$$

where $\mathbf{H}(j\omega)$ is found by substituting $s = j\omega$ in system transfer function.

$$\begin{aligned}
\text{Hence } y(t) &= \text{Re}\left\{e^{j\omega t}\mathbf{H}(j\omega)\right\} \\
&= \text{Re}\left\{e^{j\omega t} \qquad\qquad\qquad\qquad \right\} \\
&= \text{Re}\left\{|\mathbf{H}(j\omega)|e \qquad\quad ^{+\arg[\mathbf{H}(j\omega)])}\right\} \\
&= |\mathbf{H}(j\omega)|\text{Re}\left\{e \qquad\quad ^{+\arg[\mathbf{H}(j\omega)])}\right\}
\end{aligned}$$

Remember $\text{Re}[e^{j\theta}] = cos(\theta)$

$$= \left|\mathbf{H}(j\omega)\right| \cos\left(\omega t + \arg[\mathbf{H}(j\omega)]\right)$$

Remember, for any given value of ω (some frequency in radians/sec), $\mathbf{H}(j\omega)$ is just some complex number.

2.5 ALTERNATIVE DERIVATION OF BASIC RESULT FOR RATIONAL TRANSFER FUNCTIONS ONLY

$$\mathbf{H}(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)^{v_1}(s-p_2)^{v_2}\dots(s-p_n)^{v_n}}$$

Remember $\mathcal{L}(e^{-at}) = \frac{1}{s+a}$

$$x(t) = e^{j\omega t} \Rightarrow \mathbf{X}(s) = \frac{1}{s-j\omega}$$

$$\begin{aligned}\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{X}(s) &= \frac{n(s)}{d(s)} \times \frac{1}{s-j\omega} \\ &= \frac{A_0}{s-j\omega} + \frac{A_1}{(s-p_1)^{v_1}} + \dots\end{aligned}$$

By cover up rule:

$$A_0 = \frac{n(j\omega)}{d(j\omega)} = \mathbf{H}(j\omega) \quad \text{From very first line above}$$

$$\mathbf{Y}(s) = \frac{\mathbf{H}(j\omega)}{(s-j\omega)} + \frac{A_1}{(s-p_1)^{v_1}} + \dots$$

$$y(t) = \mathbf{H}(j\omega)e^{j\omega t} + \underbrace{\frac{A_1 t^{v_i-1}}{(v_i-1)!}e^{tp_i} + \dots}$$

$$= \mathbf{H}(j\omega)e^{j\omega t} \quad \text{If asymptotically stable.}$$

2.6 WATCH OUT FOR THIS

Given an input $x(t) = \sin(3t)$ to a system with transfer function $\mathbf{H}(s)$, what is the output of the system $y(t)$ when it has attained a steady state?

Typical answer :

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{X}(s) = \mathbf{H}(s)\frac{3}{s^2+9}$$
$$= \ldots \text{ Partial fractions}$$
$$\text{So } y(t) = \ldots \text{Use Inverse Laplace Transform.}$$

NOT WRONG BUT LONG!!

Faster way is to recognise that sinusoidal signals passing though LTI systems come out the other end (after transients have dies away) scaled by the gain of the system at the frequency of the input signal and lagged by some angle. SO:

$$y(t) = \left|\mathbf{H}(j3)\right| \sin(3t + \arg[\mathbf{H}(j3)])$$

Note that if the question does *not* say anything about **steady state**, or it requires you to calculate the transient output as well as the steady state output, then you cannot use this expression, but will have to work out the full response using the full Laplace domain or convolution analysis.

EXAMPLE: Given an input $x(t) = \sin(3t)$ to a system with transfer function $\mathbf{H}(s)$ (shown below), what is the output of the system $y(t)$ when it has attained a steady state?

$$\mathbf{H}(s) = \frac{s+2}{(s+4)(s+7)}$$

we know that the steady state output is going to be a sine wave at the same frequancy, but with some phase lag and some different amplitude. The general answer is

$$y(t) = \Big|\mathbf{H}(j\omega)\Big| \sin(\omega t + \arg[\mathbf{H}(j\omega)])$$

where ω is the frequency of the input signal.

because $x(t) = \sin(3t) \Rightarrow \omega = 3$ Radians per sec (rad sec^{-1})

so answer is

$$y(t) = \Big|\mathbf{H}(j3)\Big| \sin(3t + \arg[\mathbf{H}(j3)])$$

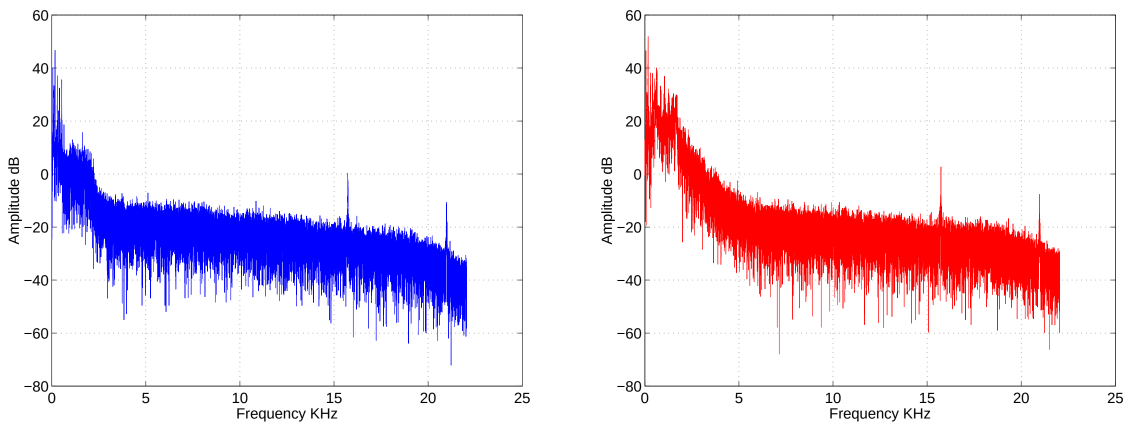
so now need to work out $\mathbf{H}(j3)$.

$$\begin{aligned} \mathbf{H}(3j) &= \frac{3j+2}{\qquad} \\ \Rightarrow |\mathbf{H}(3j)| &= \left|\frac{3j+2}{\qquad}\right| \\ &= \frac{|3j+2|}{|(3j+4)||(3j+7)|} = \frac{\sqrt{3^2+2^2}}{\sqrt{3^2+4^2}\sqrt{\qquad}} \\ &= 0.95 \\ \arg(\mathbf{H}(3j)) &= \arg\left[\frac{3j+2}{(3j+4)(3j+7)}\right] = -.0656\text{rad} \end{aligned}$$

3 The problem again

We now know, given a system transfer function, how to calculate its frequency response to a given input sinusoid at a particular frequency.

Take a look at our corrupted signal again. The plot on the left shows the frequency content (amplitude in dB) just before the speech starts, and on the right is the content just after the speech starts. Before the speech starts, there is only noise and hum, so comparing the two plots tells us where the frequencies of the hum (mostly) lie. You can see that sadly, the hum occupies a range of low frequencies, not just one. We'd need to design a system to suppress all of these frequencies to make a start at our system design.



Left: Frequency content of first second of recording, Right: frequency content of recording once speech has started

At the moment we only know how to predict the frequency response of a system for a single sinusoidal frequency. This is not good enough for our design. We need to be able to summarise the frequency reponse over a broad range in some way. For this we need to be able to plot gain and phase response as a function of frequency. Then we can try to adjust things to knock out our *hum*.

BODE DIAGRAMS

- Need to summarize the frequency response of a system in a useful way. This is done with plots of Gain and Phase Vs. frequency. (As was shown earlier with examples of response of microphone etc.)
- A Bode diagram of a system consists of 2 graphs GAIN (log) Vs FREQUENCY (Log) and PHASE (lin) vs FREQUENCY (log).
- Why? Because BOTH amplitude and phase of a signal changes as it passes through the system.
- Gain is $\left|\mathbf{H}(j\omega)\right|$ and Phase is $\arg[\mathbf{H}(\jmath\omega)]$.
- Gain is plotted in DECIBELS $= 20 \times \log_{10}\left|\mathbf{H}(j\omega)\right|$ on a linear scale.
- Phase is plotted in Radians or Degrees on a linear scale.
- BOTH PLOTS ARE MADE AGAINST LOG FREQUENCY. EASIEST TO USE LOG PAPER IF POSSIBLE
- Log Frequency is used to allow a large range of frequency to be handled on the same plot.
- By using Log Frequency, drawing Bode diagrams is made simpler than if we had to draw the curves on a linear scale. Also, as our perception of signals obeys log-like frequency laws, it all makes sense to view the system frequency response in a similar way.

- These days, lots of packages like **Matlab** allow you to plot Bode diagrams automatically. The Matlab function to do this is bode(<system transfer function>). Its that easy.
- So why the hell are you going to learn about bode plots?
- Its because you need to interpret these plots, and for design, you need to get a handle on how to change your transfer function to get the desired effect.

A really important table

Gain	0.1	$\frac{1}{\sqrt{2}}$	1	10	100
	10^{-1}	$10^{-.1505}$	10^0	10^1	10^2
$\text{dB} = 20\log_{10}(\text{Gain})$	-20	-3.01	0	20	40

- A gain of ZERO dBs means a gain of UNITY ! (no change to amplitude of signal as it passes throgh the system)
- A gain of -ve dBs means attenuation ! (amplitude of signal reduces as it passes through the system)
- A gain of +ve dBs means amplification ! (amplitude of signal increases as it passes through the system)

4 Bode Diagrams

Main idea is to factorise Transfer Function.

$$\begin{aligned}
\mathbf{H}(s) &= \frac{(s+1.5)(1-\frac{s}{20})}{(s+5)(1+s+25s^2)} \\
&= \frac{3}{10} \times \frac{1}{1+s+25s^2} \times (1+\frac{2}{3}s) \\
&\times \frac{1}{1+\frac{1}{5}s} \times (1-\frac{1}{20}s)
\end{aligned}$$

AMPLITUDE GAIN :

$$\begin{aligned}
20\log_{10}|\bar{h}(j\omega)| &= 20\log_{10}\left(\left|\frac{3}{10}\right|\right) + 20\log_{10}\left(\left|\frac{1}{1+j\omega+25(j\omega)^2}\right|\right) \\
&+ 20\log_{10}\left(\left|1+\frac{2}{3}j\omega\right|\right) + 20\log_{10}\left(\left|\frac{1}{1+\frac{1}{5}j\omega}\right|\right) \\
&+ 20\log_{10}\left(\left|1-\frac{1}{20}j\omega\right|\right) \\
&= 20\log_{10}(0.3) - 20\log_{10}\left|1-25\omega^2+j\omega\right| \\
&+ 20\log_{10}\left|1+\frac{2}{3}j\omega\right|
\end{aligned}$$

$$\begin{aligned}
&= 20\log_{10}(0.3) - 20\log_{10}\sqrt{(1-25\omega^2)^2+\omega^2} \\
&+ 20\log_{10}\sqrt{1+\frac{4}{9}\omega^2}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{H}(s) &= \frac{(s+1.5)(1-\frac{s}{20})}{(s+5)(1+s+25s^2)} \\
&= \frac{3}{10} \times \frac{1}{1+s+25s^2} \times (1+\frac{2}{3}s) \\
&\times \frac{1}{1+\frac{1}{5}s} \times (1-\frac{1}{20}s)
\end{aligned}
$$

PHASE CHANGE:

$$
\begin{aligned}
\arg[\mathbf{H}(j\omega)] &= \arg\left(\frac{3}{10}\right) + \arg\left(\frac{1}{1+j\omega+25(j\omega)^2}\right) \\
&+ \arg\left(1+\frac{2}{3}j\omega\right) + \arg\left(\frac{1}{1+\frac{1}{5}j\omega}\right) \\
&+ \arg\left(1-\frac{1}{20}j\omega\right) \\
&= 0 - \arg\left[(1-25\omega^2)+j\omega\right] + \arg(1+\frac{2}{3}j\omega) + \ldots \\
&= 0 - \tan^{-1}\left(\frac{\omega}{1-25\omega^2}\right) + \tan^{-1}\left(\quad\right) \ldots
\end{aligned}
$$

4.1 How to draw the curves

- Have just seen that we get LOG(GAIN) by adding LOG(MAGNITUDES). We get PHASE change by adding ARGUMENTS.
- Can now treat each term separately (you will meet only a few types of terms). TRICK to fast drawing is to use ASYMPTOTES.

4.2 WHAT IS BODE PLOT OF $(1 + sT)$?

Replace s by :

$$20\log_{10}\left|1 + j\omega T\right| = 20\log_{10}\sqrt{1+\omega^2T^2}$$

$$= \begin{cases} \approx 0 & \text{dB for} \\ = 20\log_{10}\sqrt{2} & = 3\text{dB for} \\ \approx 20\log_{10}(\omega T) & \text{dB for} \end{cases}$$

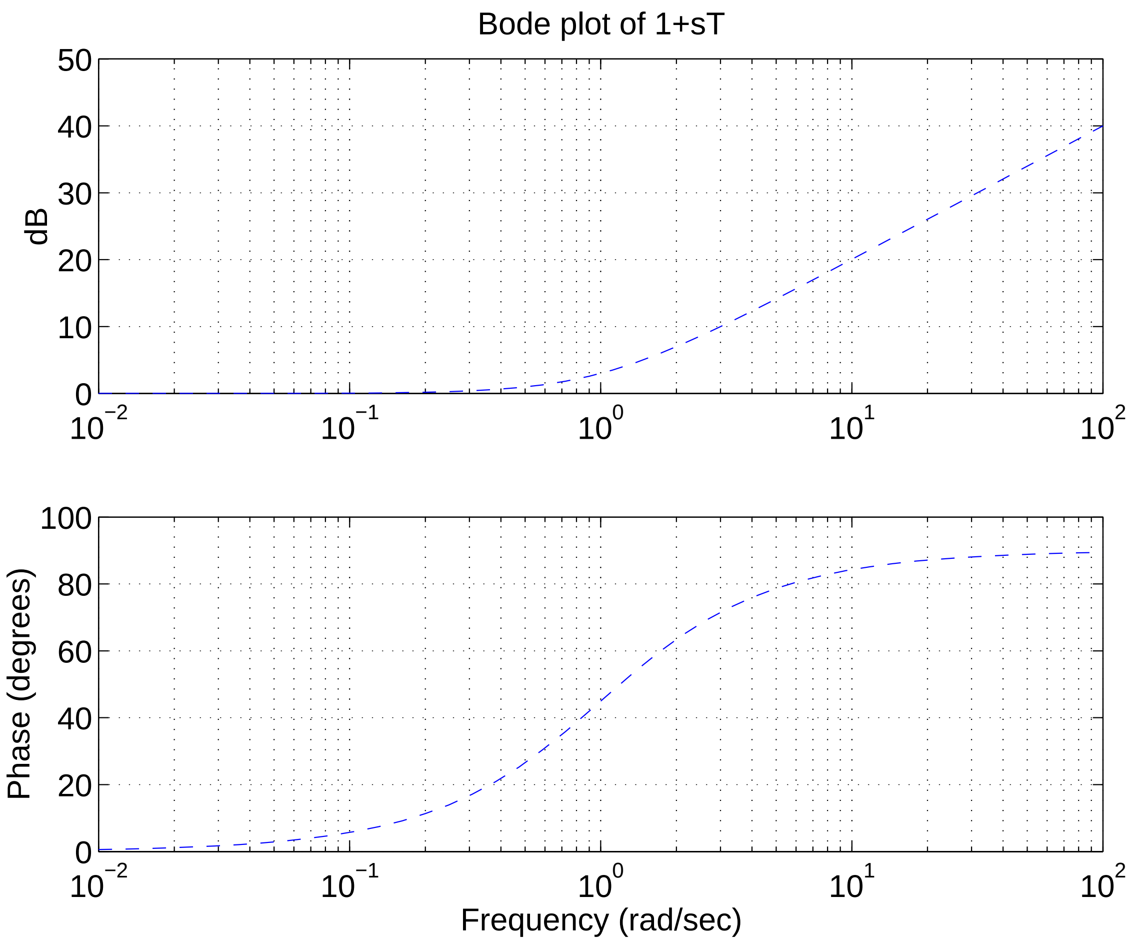
$$\arg[1 + j\omega T] = \tan^{-1}(\omega T)$$

$$= \begin{cases} \approx 0 & \text{for} \\ = 45^\circ \text{ (or } \pi/4 \text{ rad)} & \text{for} \\ \approx 90^\circ \text{ (or } \pi/2 \text{ rad)} & \text{for} \end{cases}$$

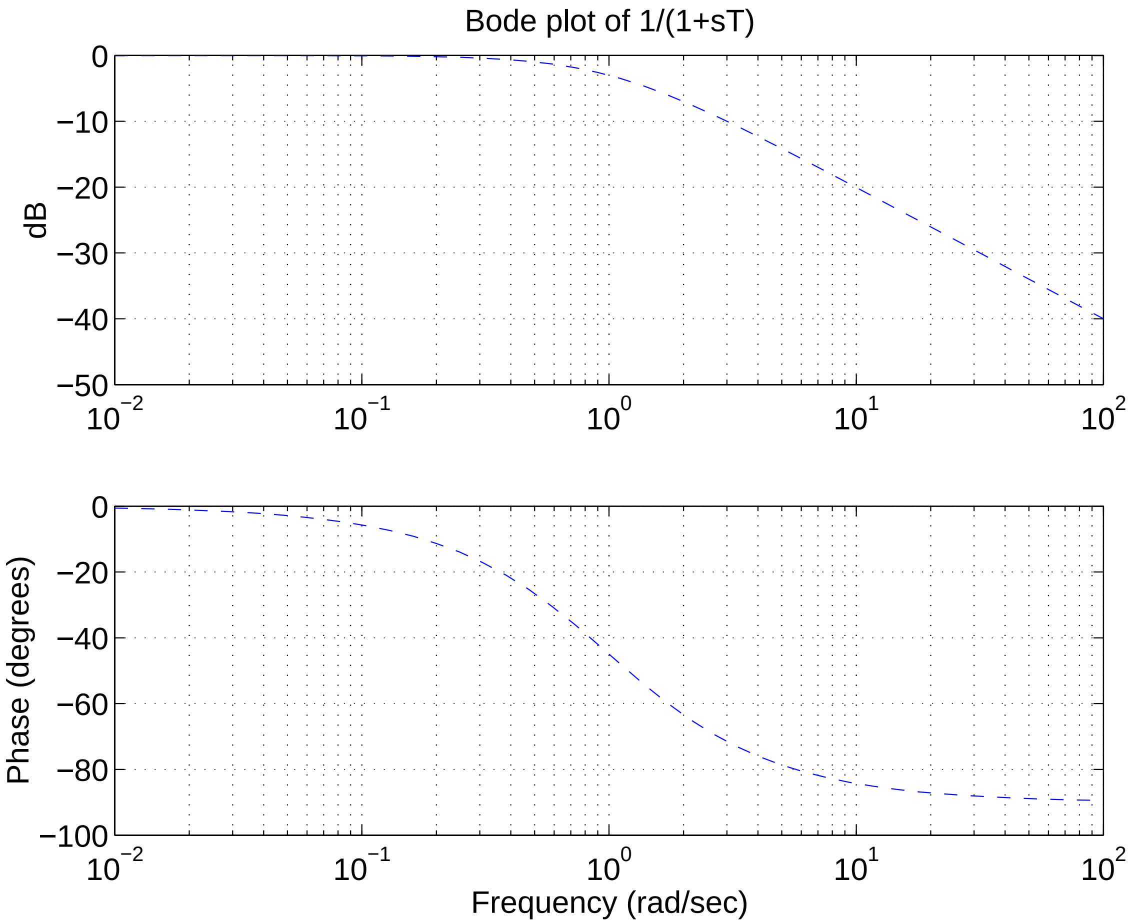
$\frac{1}{T}$ is called a 'breakpoint', or 'break frequency' or 'corner frequency' of the asymptote.

PLOTTING BODE DIAGRAM OF $(1 + sT)$

1. Mark point at break frequency $\omega = \frac{1}{T}$, and log(gain) = 3 dB.
2. For $\omega << \frac{1}{T}$ the straight line gain asymptote is a horizontal line of height 0 dB.
3. For $\omega >> \frac{1}{T}$, draw straight line gain asymptote whose gradient is 20 db per Decade. Reason:
   Consider $\omega = \frac{1}{T}$, log(gain of asymptote) $= 20\log_{10}(\omega T) = 20\log_{10}(1)$ $= 0$ dB.
   Now consider $\omega = \frac{10}{T}$, log(gain of asymptote) $= 20\log_{10}(\omega T) =$ $20\log_{10}(10) = 20$ dB.
   So for an increase in frequency of a FACTOR of 10 (a decade !!) the asymptote changes in gain by 20dB. So its slope is 20dB per decade. **Two frequencies ω_1 and ω_2 are separated by one DECADE if** $\frac{\omega_1}{\omega_2} = 10$.
   THIS IS VERY IMPORTANT TO REMEMBER BECAUSE A DECADE IS THE DISTANCE BETWEEN 1,10; 10,100; 100,1000, AND YOU ARE PLOTTING LOG(GAIN) (Y-AXIS) VS LOG(FREQUENCY) (X-AXIS). ON LOG PAPER THESE DISTANCES (1,10; 10,100 ETC) ARE THE SAME.


Bode plot of 1+sT
dB
50
40
30
20
10
0
Phase (degrees)
100
80
60
40
20
0
10^−2
10^−1
10^0
10^1
10^2
Frequency (rad/sec)

4.3 WHAT IS BODE PLOT OF $1/(1+sT)$?



$$-20\log_{10}\left|1+j\omega T\right| = -20\log_{10}\sqrt{1+\omega^2T^2}$$

$$= \begin{cases} \approx 0 & \text{dB for } \omega << (1/T) \\ = -20\log_{10}\sqrt{2} & = -3\text{dB for } \omega = (1/T) \\ \approx -20\log_{10}(\omega T) & \text{dB for dB for } \omega >> (1/T) \end{cases}$$

$$-\arg[1+j\omega T] = -\tan^{-1}(\omega T)$$

$$= \begin{cases} \approx 0 & \text{for } \omega << (1/T) \\ = -45^\circ \text{ (or } -\pi/4 \text{ rad)} & \text{for } \omega = (1/T) \\ \approx -90^\circ \text{ (or } -\pi/2 \text{ rad)} & \text{for } \omega >> (1/T) \end{cases}$$

BODE PLOT OF $\frac{1}{s}$

Put $s =$.

$$\frac{1}{s} = \frac{1}{j\omega}$$

GAIN

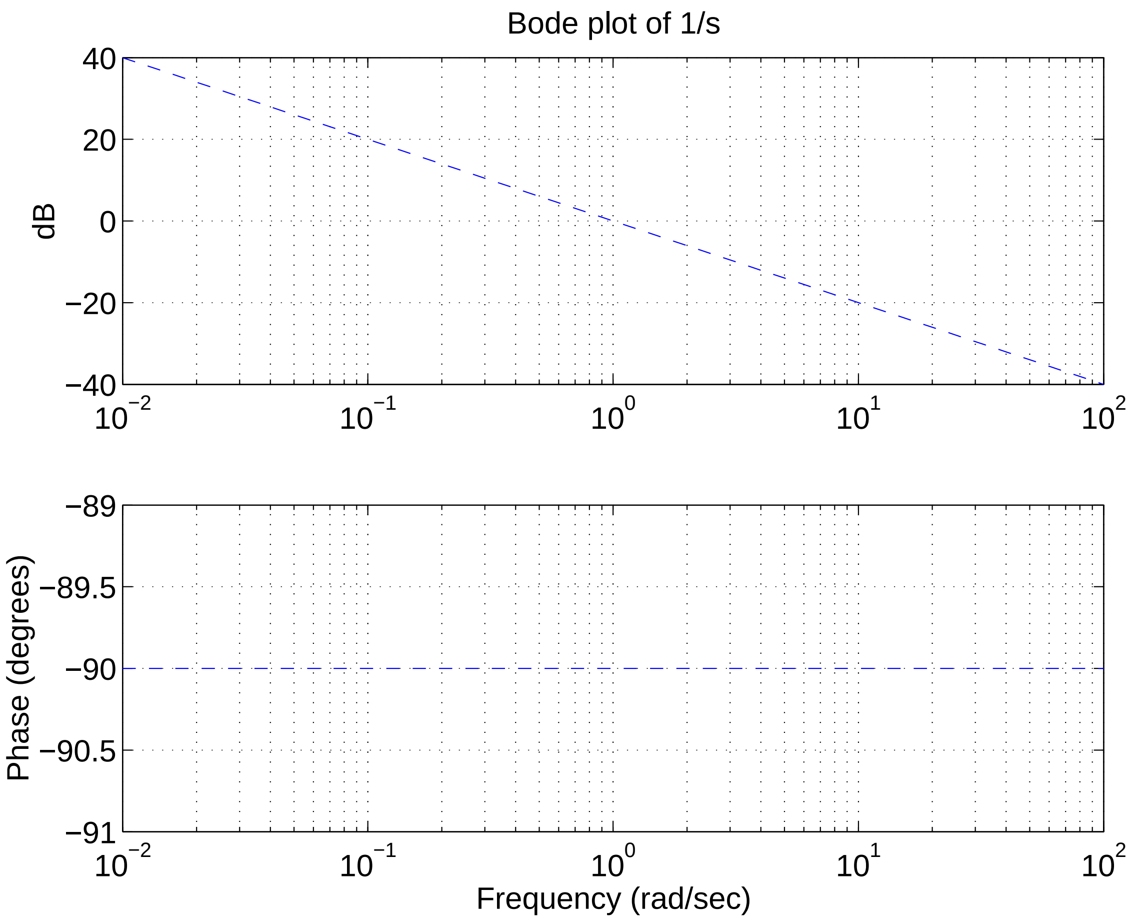
$$20\log_{10}\left|\frac{1}{j\omega}\right| = -20\log_{10}(\omega) \text{ dB}$$

PHASE

$$\arg\left[\frac{1}{j\omega}\right] = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) = -\tan^{-1}(\infty) = -90°$$

1. The Gain asymptote is just a straight line with gradient -20 dB per DECADE. There is no break frequency, its simple enough to draw.

2. Reason :
   Consider $\omega = 0.1$ rad/sec, log(gain of asymptote) $= -20\log_{10}(0.1)$ $= 20$ dB.
   Consider $\omega = 10$ rad/sec, log(gain of asymptote) $= -20\log_{10}(10)$ $= -20$ dB.
   So its a straight line of gradient -20 dB per decade. Note that it passes through the point $(1, 0)$ Because when $\omega = 1$, gain $= 0$.


Bode plot of 1/s
dB
40
20
0
−20
−40
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Phase (degrees)
−89
−89.5
−90
−90.5
−91
Frequency (rad/sec)

4.4 BODE PLOT OF $(1 + 2c\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})$

Replace s by : $j\omega$

$$
\begin{aligned}
20\log_{10}\left|1 + 2c\frac{j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2}\right| &= 20\log_{10}\left|(1 - \frac{\omega^2}{\omega_n^2}) + j2c\frac{\omega}{\omega_n}\right| \\
&= 20\log_{10}\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2c\frac{\omega}{\omega_n}\right)^2}
\end{aligned}
$$

$$
= \begin{cases} \approx 0 \text{ dB for } \omega << \omega_n \\ = 20\log_{10}(2c) \text{ dB for } \omega = \omega_n \\ \approx 40\log_{10}\left(\frac{\omega}{\omega_n}\right) \text{ dB for } \omega >> \omega_n \end{cases}
$$

$$
\begin{aligned}
\arg\left[1 + 2c\frac{j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2}\right] &= \arg\left[(1 - \frac{\omega^2}{\omega_n^2}) + j2c\frac{\omega}{\omega_n}\right] \\
&= \tan^{-1}\left(\frac{2c\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)
\end{aligned}
$$

$$
= \begin{cases} \approx 0^\circ \text{ for} \\ = 90^\circ \text{ for} \\ \approx 180^\circ \text{ for} \end{cases}
$$

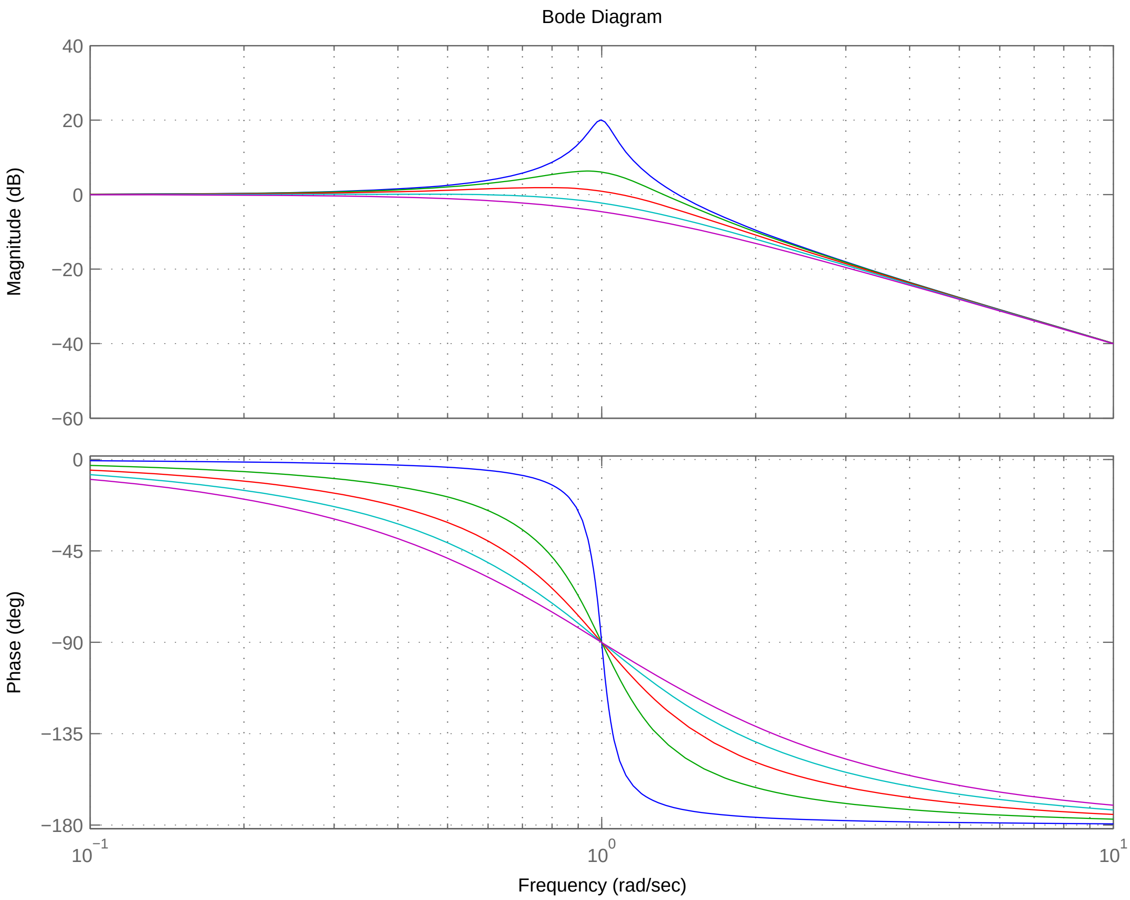
PLOTTING BODE DIAGRAM OF $\frac{1}{(1+2c\frac{s}{\omega_n}+\frac{s^2}{\omega_n^2})}$

1. Mark point at break frequency $\omega = \omega_n$, and $\log(\text{gain}) = -20\log_{10}(2c)$ dB. (NEGATIVE BECAUSE THE FUNCTION IS 1/(The thing we just looked at on the previous page).)

2. For $\omega << \omega_n$ the straight line gain asymptote is a horizontal line of height 0 dB.

3. For $\omega >> \omega_n$, draw straight line gain asymptote whose gradient is -40 db per Decade. Reason:
   Consider $\omega = \omega_n$,
   $\log(\text{gain of asymptote}) = -40\log_{10}(\frac{\omega}{\omega_n}) = -40\log_{10}(1) = 0$ dB.
   Now consider $\omega = 10\omega_n$,
   $\log(\text{gain of asymptote}) = -40\log_{10}(\frac{\omega}{\omega_n}) = -40\log_{10}(10) = 40$ dB.
   So for an increase in frequency of a FACTOR of 10 (a decade) the asymptote changes in gain by -40dB. So its slope is -40dB per decade. **Two frequencies ω_1 and ω_2 are separated by one DECADE if** $\frac{\omega_1}{\omega_2} = 10$.

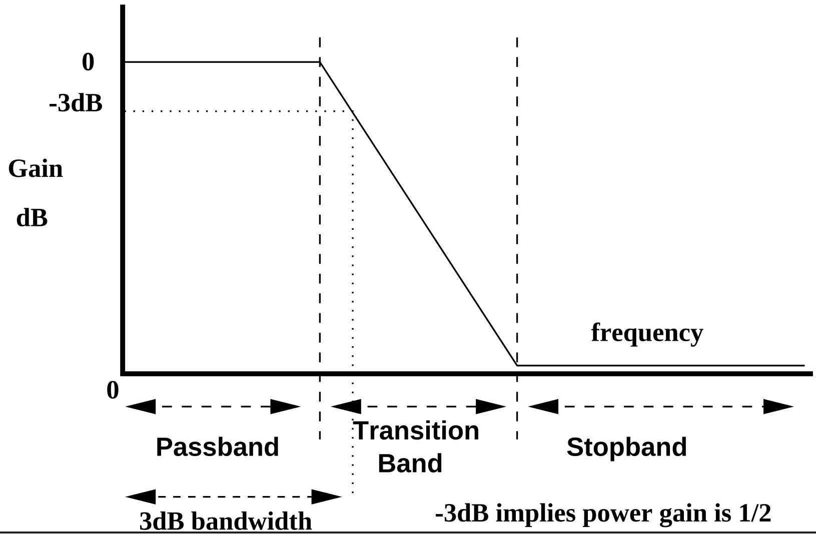
4. Note that this high frequency asymptote does indeed pass through the point $(\omega_n, 0)$ on the Bode diagram.

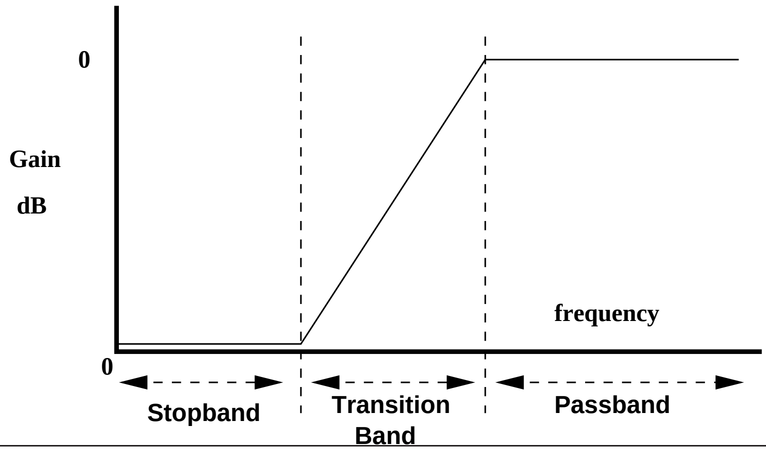

Bode Diagram
Magnitude (dB)
40
20
0
−20
−40
−60
Phase (deg)
0
−45
−90
−135
−180
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

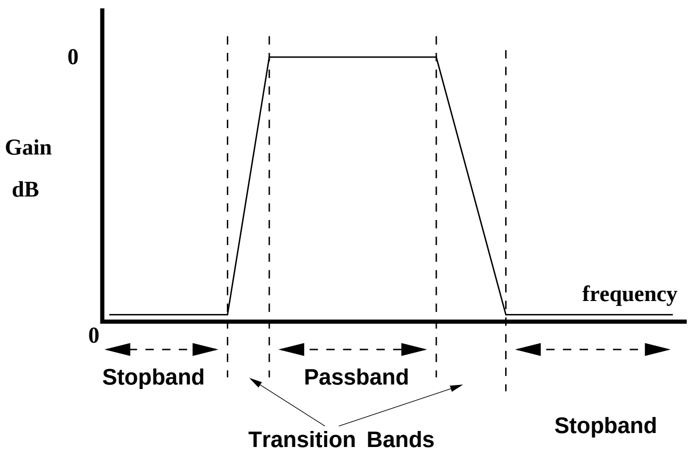
5 FILTERS

- Filters are systems which are designed specifically for altering the frequency content of signals.
- A *Low Pass Filter* allows low frequencies to pass through the system unaffected but it stops or 'attenuates' high frequencies from passing through the system. The output signal therefore contains either no high frequencies or high frequencies of a smaller amplitude than compared to the input signal. What is 'low' or 'high' depends on the requirements of the designer.
- A *High Pass Filter* allows **high** frequencies to pass through the system unaffected but it **stops or 'attenuates' low** frequencies from passing through the system. The output signal therefore contains either no low frequencies or low frequencies of a smaller amplitude than compared to the input signal. What is 'low' or 'high' depends on the requirements of the designer.
- A *Band Pass Filter* allows a mid-range of selected frequencies to pass through the system unaffected, and attenuates or 'stops' all other frequencies outside this range.
- The range of frequencies which is allowed to pass through the filter unaffected is called the 'pass band' of the filter. The range which is attenuated is called the 'stop band'.
- These systems are called 'filters' because they affect some frequencies differently from others. This is just like the action of filter paper being used to separate a mixture of particles of different sizes, water from sand say.
- Remember our problem: to make a system to zap some frequencies in a corrupted signal that we think contain the hum? Now we know what we need is to design a **filter** to do the trick.

5.1 Generic Gain of different types of filters







5.2 An important filter: THE BUTTERWORTH FILTER

- You can spend alot of time testing functions of s to see what kind of filters you can come up with. In this course we will just look at one popular form.
- The butterworth filter is 'maximally' flat in the passband.
- An Nth order lowpass Butterworth filter has a frequency response with a form as follows:

$$\left|G(j\omega)\right|^2 = \frac{1}{1+(\frac{\omega}{\omega_c})^{2N}}$$

So the second order ($N = 2$) low-pass Butterworth filter has a frequency response of the form

$$\left|G(j\omega)\right|^2 = \frac{1}{1+(\frac{\omega}{\omega_c})^{4}}$$

- Regardless of the value of N the filter gain always has a value of 1 at $\omega = 0$ and passes through $\sqrt{2}/2$ at $\omega = \omega_c$, the 3dB Cutoff frequency.
- It can be made using op-amps in several ways. One example is shown below in which the op-amp is assumed to have infinite gain.
- We will now work through an entire analysis of this filter including deriving the transfer function and then the frequency response. We will then define the *bandwidth* of the filter.

THE BUTTERWORTH FILTER EXAMPLE

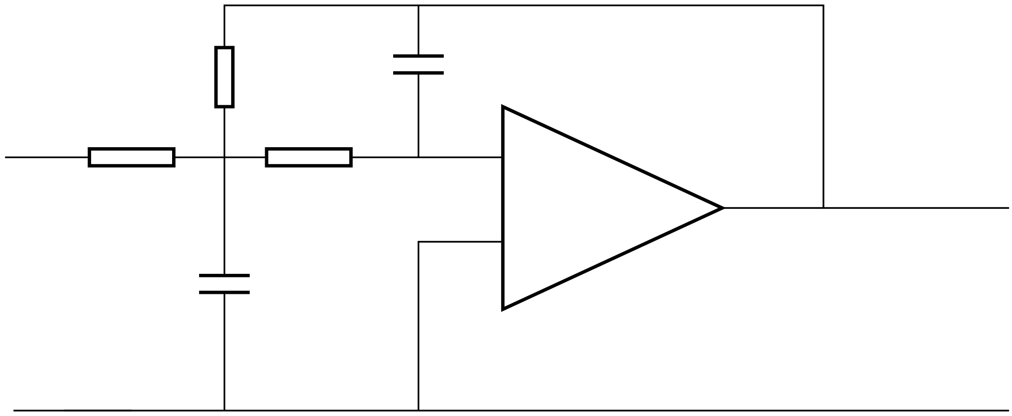


Figure 1: A Low Pass Filter Network. Suitable choices for the components will result a low pass Butterworth filter.

$R_1 = R_2 = R_3 = 1\Omega$, $C_1 = 3/\sqrt{2}F$, $C_2 = \sqrt{2}/3F$ First, find the transfer function V_o/V_i. (Dropping the (s) arguments for brevity).

Sum currents at A

$$\frac{\mathbf{V}_i - \mathbf{V}_A}{R_1} = sC_1\mathbf{V}_A + \frac{\mathbf{V}_A}{R_2} + \frac{\mathbf{V}_A - \mathbf{V}_o}{R_3}$$

Substituting component values gives

$$\Rightarrow \mathbf{V}_i - \mathbf{V}_A = \frac{3}{\sqrt{2}}s\mathbf{V}_A + \mathbf{V}_A + \mathbf{V}_A - \mathbf{V}_o \quad (1)$$

Sum currents at B

$$\frac{\mathbf{V}_A}{R_2} = -\frac{\mathbf{V}_o}{1/sC_2} = -\mathbf{V}_o\frac{s\sqrt{2}}{3} \quad (2)$$

$$\text{From 1} \quad \mathbf{V}_i = \mathbf{V}_A\left[s\frac{3}{\sqrt{2}} + 3\right] - \mathbf{V}_o$$

Subst 2 into 1 gives

$$\mathbf{V}_i = \frac{-\sqrt{2}s\mathbf{V}_o}{3}\left[s\frac{3}{\sqrt{2}} + 3\right] - \mathbf{V}_o$$

$$= \mathbf{V}_o\left[-s^2 - \sqrt{2}s - 1\right]$$

$$\Rightarrow \quad \mathbf{H}(s) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \frac{-1}{s^2 + \sqrt{2}s + 1}$$

Frequency response: put $s = j\omega$. Is it a Butterworth filter?

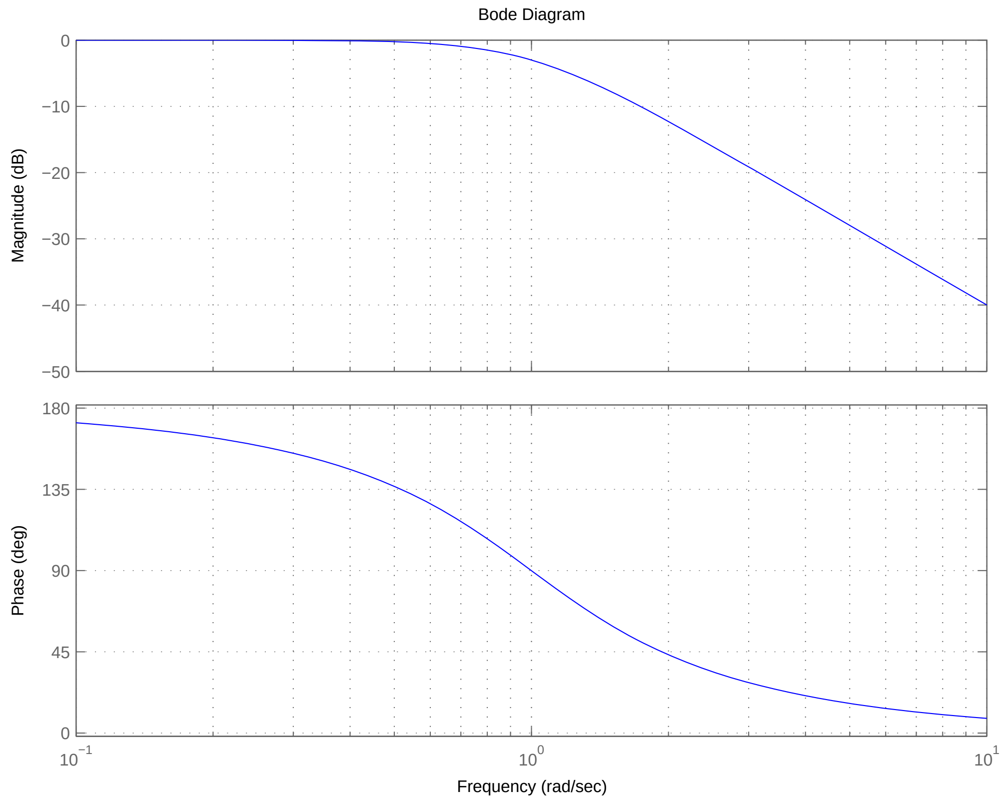
$$\begin{aligned}
\left|\mathbf{H}(j\omega)\right| &= \left|\frac{-1}{-\omega^2 + \sqrt{2}j\omega + 1}\right| \\
\left|\mathbf{H}(j\omega)\right|^2 &= \frac{1}{(1-\omega^2)^2 + 2\omega^2} \\
&= \frac{1}{1 - 2\omega^2 + \omega^4 + 2\omega^2} \\
&= \frac{1}{1+\omega^4} \\
&= \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^4}
\end{aligned}$$

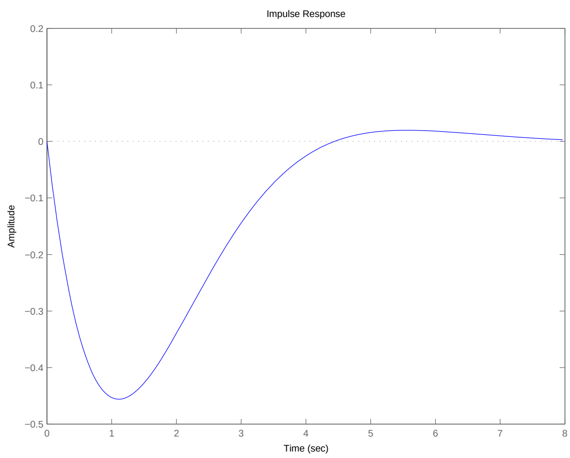
where $\omega_c = 1$. So its a Butterworth filter with $\omega_c = 1$ rad/sec.

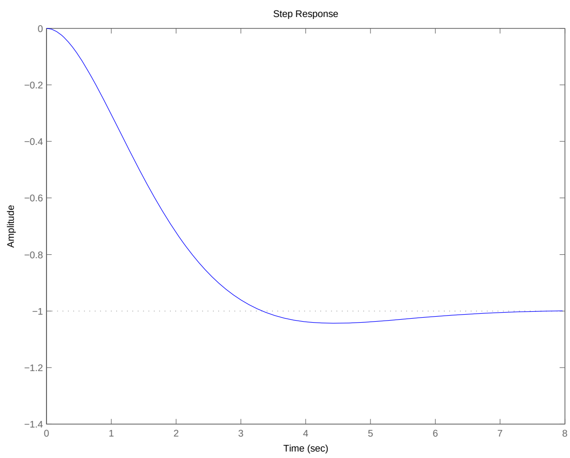
3dB bandwidth is the range of frequency over which the gain is > -3dB.

$$\begin{aligned}
20\log_{10}\left|\mathbf{H}(j\omega_{3db})\right| &> -3 \\
\Rightarrow 20\log_{10}\left(\frac{1}{\sqrt{(1-\omega_{3db}^2)^2 + 2\omega_{3db}^2)}}\right) &> -3 \\
\Rightarrow \frac{1}{\sqrt{(1-\omega_{3db}^2)^2 + 2\omega_{3db}^2)}} &> \frac{1}{\sqrt{-2}} \\
\Rightarrow \sqrt{(1-\omega_{3db}^2)^2 + 2\omega_{3db}^2)} &< 2 \\
\Rightarrow 1 + \omega_{3db}^4 &< 2 \\
\Rightarrow \omega_{3db}^4 &< 1
\end{aligned}$$

So 3dB bandwidth of filter is $\omega_{3db} = 1$ rad/sec. That means that this filter has a gain GREATER than $-3dB$ for signals with a frequency from 0 to 1 rad/sec. Above this frequency, the filter *attenuates* the signal.


Bode Diagram
Magnitude (dB)
0
−10
−20
−30
−40
−50
Phase (deg)
180
135
90
45
0
10^−1
10^0
10^1
Frequency (rad/sec)


Impulse Response
Amplitude
Time (sec)


Step Response
Amplitude
Time (sec)

6 POLE-ZERO DIAGRAM AND THE FREQUENCY RESPONSE

$$\mathbf{H}(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)}$$

$$\mathbf{H}(j\omega) =$$

$$\left|\mathbf{H}(j\omega)\right| \propto \frac{\text{PRODUCT OF DISTANCES FROM } j\omega \text{ TO ZEROS}}{\text{PRODUCT OF DISTANCES FROM } j\omega \text{ TO POLES}}$$

THEREFORE a zero near $j\omega$ axis $\Rightarrow$
AND a pole near $j\omega$ axis $\Rightarrow$