SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 6.
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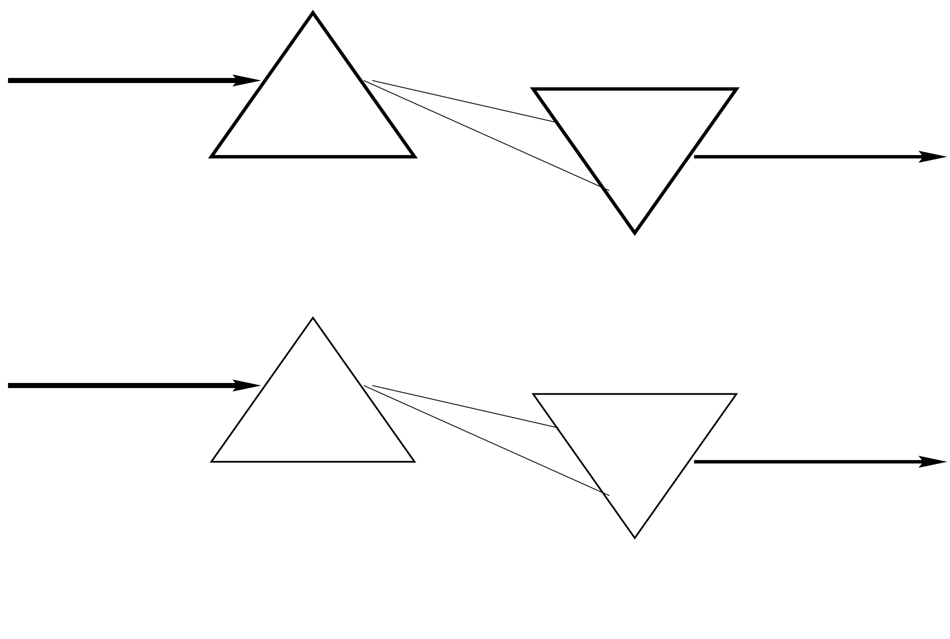
FOURIER ANALYSIS

- Have seen how the behaviour of systems can be represented in terms of their frequency response. Now want to consider frequency content of *signals*.
- Fundamental idea is that any signal can be represented as a sum of sines and cosines of different amplitudes and frequency.
- This representation can be thought of as decomposing a signal into sinusoidal components.
- There is a good analogy with the effect of a prism on white light. The prism decomposes the light into its various 'coloured' components. This sequence of components is called the 'spectrum' of the light and another differently oriented prism can recombine these 'spectral components' to regenerate the original light beam.



- Representing a signal in terms of a sum of Sines and Cosines is a good idea because
  - We have already seen that sines and cosines pass through an LTI system almost unchanged except for amplitude and phase. (They are eigenfunctions of LTI systems.) So if we know how to sythesize a signal from a bunch of sines and cosines then we can always tell what any LTI system will do to any signal.
  - the individual spectral components of the signal often make the nature of the signal clearer e.g. speech recognition.
  - The human perception of many signals (e.g. audio and video) can be directly related to the spectral components of these signals. There are indeed very sophisticated audio and video 'filters' in your head.
- Fourier discovered (1807) that we can decompose practical signals into a sum of trig. functions. Conversely we can synthesize a practical waveform or signal by adding together a number of these functions.
- In theory it may be necessary to add an infinite number of them in order to synthesize the signal perfectly. In practice we must work with only a *finite* number so producing a signal which is an *approximation* to the true shape.
- Before becoming quantitative, lets talk in general terms.

1 General view of Fourier analysis for Periodic Signals

- Consider a sawtooth wave (important signal used as timebase waveforms for TV and oscilloscopes to control the horizontal deflection of the 'flying spot'.)

- The signal is periodic therefore the sinusoidal waves needed to synthesize it are harmonically related. This means that their frequencies bear a simple integer relationship to each other.

- Using formulae which we consider later on, the waveform can be written:

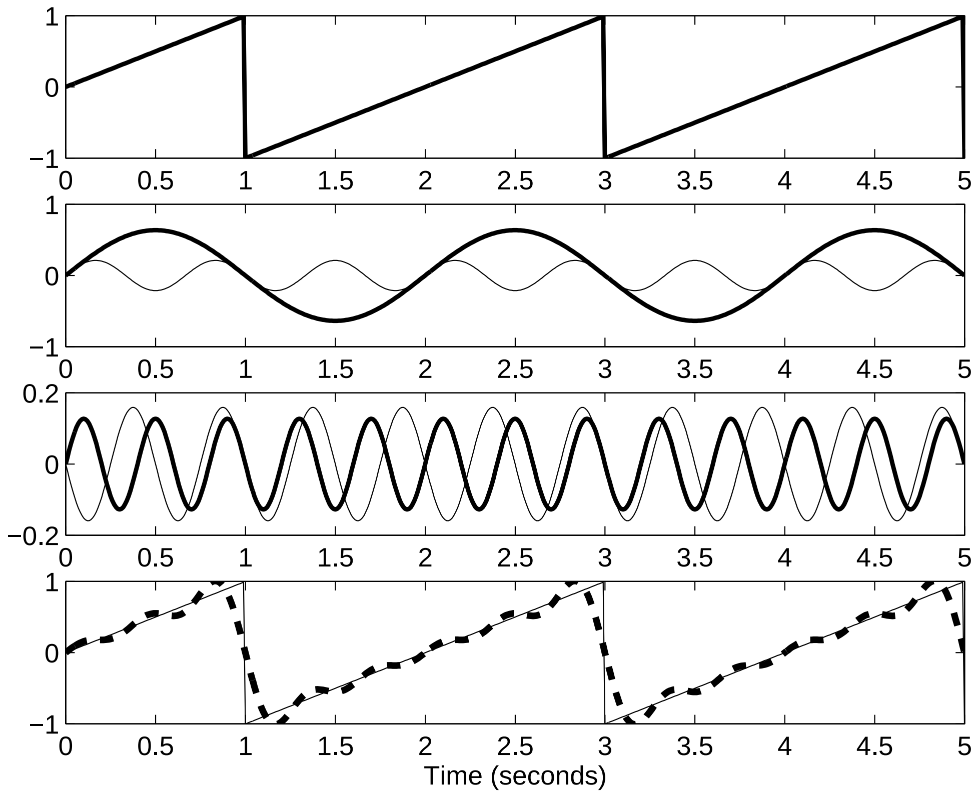
$$x(t) = \frac{2}{\pi}\sin(\omega_0 t) - \frac{1}{\pi}\sin(2\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) - \frac{1}{2\pi}\sin(4\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t) + \ldots$$

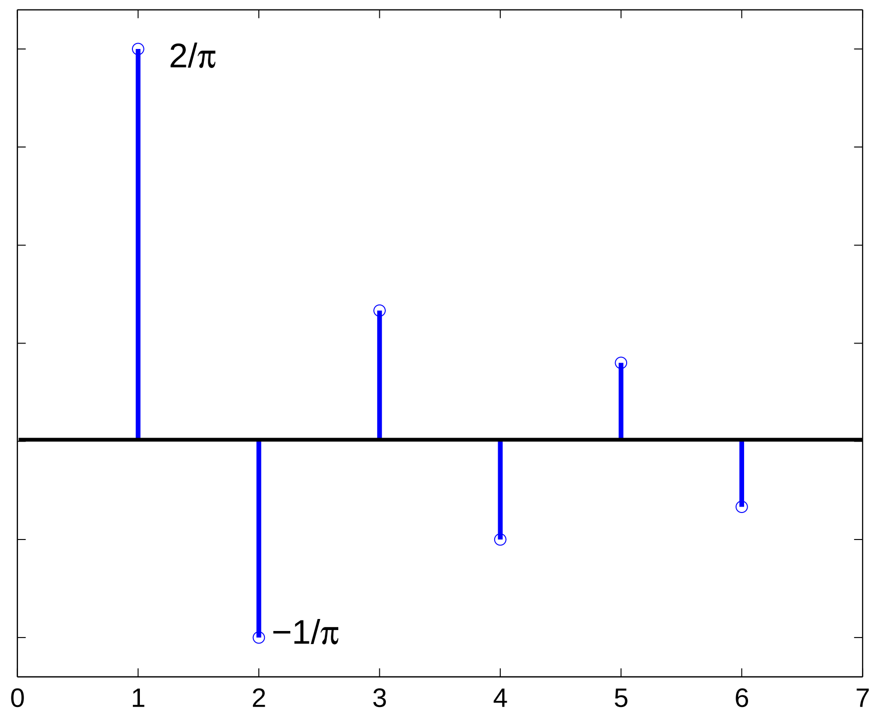
- It contains a component with a fundamental frequency ω_0 which has the same period as $x(t)$ itself and a bunch of harmonics at $2\omega_0$ (2nd harmonic), $3\omega_0$ (3rd harmonic) etc.

- Note that $x(t)$ is an odd function so we only need Sinusoidal waveforms to be added together to synthesize it. (Sines are ODD signals because $\sin(t) = -\sin(-t)$) .

- There are an infinite number of harmonics, but if we sum just the first four of them then we get an *approximation* to $x(t)$.

- This information can be summarized graphically by the frequency spectrum of $x(t)$ (called $X(\omega)$).





- This type of spectrum is called a 'line spectrum' because it contains a number of distinct frequency components.
- In this case the phase relationship between the components are simple

and we can draw $X(\omega)$ on one graph. In general we would need to use two graphs one for $|X(\omega)|$ and one for $\arg[X(\omega)]$. (c.f. Bode diagrams).

- Note that the sawtooth has sharp edges. We are trying to represent a signal with discontinuities (the sharp edges) by using the sum of signals (sines) with **no** discontinuities. This means that we will need to add *many* sines and together to synthesize $x(t)$ exactly. For signals with no discontinuities we would need fewer sines and cosines to get an exact representation.
- Fortunately, being Engineers, we recognise that even if we have to work with a finite number of terms, quite often the approximation for discontinuous signals is not bad at all and still quite usable. So we do not get upset by this.
- This problem did upset Lagrange however, and he rejected Fourier's paper in 1807 when it was submitted for consideration. (Laplace supported Fourier though, and Fourier published his work 15 years later anyway.)
- THIS IS AN IMPORTANT OBSERVATION. **SIGNALS** with discontinuities in them have more frequency components in their spectrum than do smooth signals. In other words, signals with discontinuities have a larger *bandwidth* than do smooth signals.

2 ORTHOGONALITY

- Another good reason for using Sines and Cosines is that these signals form an 'orthogonal basis'. To examine what this means, consider the approximation of vector quantities.

- Most of you are familiar with representing force, velocity as a vector. Suppose we have two vectors $\mathbf{v}_1$, $\mathbf{v}_2$. We may define the component of $\mathbf{v}_1$ along $\mathbf{v}_2$ as follows.

- So if we are trying to approximate $\mathbf{v}_1$ by another vector *in the direction of* $\mathbf{v}_2$, the error in the approximation is $\mathbf{v}_e$.

- The best approximation is obtained when C_{12} is chosen to make $\mathbf{v}_e$ as small as possible; hence $\mathbf{v}_e$ is perpendicular to $\mathbf{v}_2$ for the **best** approximation.

- We then say that the *component* of $\mathbf{v}_1$ along $\mathbf{v}_2$ is $C_{12}\mathbf{v}_2$. If $C_{12} = 0$ then there is **no** component along $\mathbf{v}_2$ and the vectors are orthogonal.

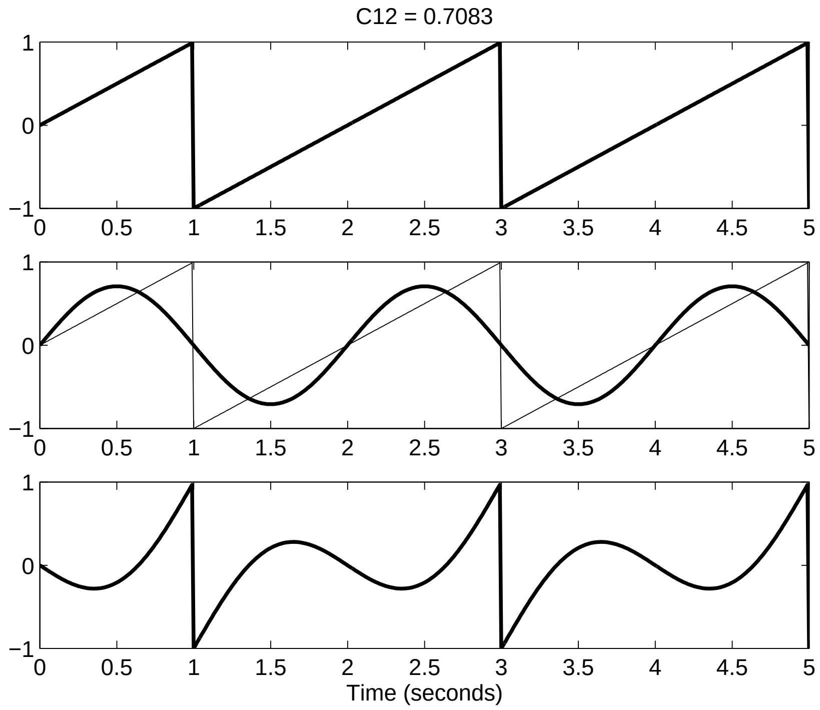
- If c_{12} is 1 and $\mathbf{v}_e = 0$; then $\mathbf{v}_1 = \mathbf{v}_2$ in both magnitude and direction.
- The amount of $\mathbf{v}_1$ along $\mathbf{v}_2$ is given by the dot product

$$C_{12} = \mathbf{v}_1.\mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2|\cos(\theta) = \sum_{k=1}^{2} \mathbf{v}_1^{(k)}\mathbf{v}_2^{(k)} \tag{1}$$

- In general $< \mathbf{v}_1.\mathbf{v}_2 >= \sum_{k=1}^{N} \mathbf{v}_1^{(k)}\mathbf{v}_2^{(k)}$

2.1 ORTHOGONALITY AND SIGNALS

- Similar ideas apply to signals
- Suppose we want to approximate $x_1(t)$ over the interval $t_1 < t < t_2$ by some other signal $x_2(t)$. In the context of Fourier analysis we can think of $x_1(t)$ as any signal and $x_2(t)$ as a sinusoidal (or complex exponential) waveform at a particular frequency.
- So we write $x_1(t) = C_{12}x_2(t) + x_e(t)$ for $t_1 < t < t_2$.
- $x_e(t)$ is the error in the approximation; C_{12} is the *amount* of $x_2(t)$ in $x_1(t)$.



- We need to minimise $x_e(t)$ by adjusting C_{12}. That is to say we want to make the approximation as good as we can, so we need to make the error as small as we can.
- $x_e(t)$ will vary over $t_1 < t < t_2$. We have to come up with some way of saying what 'error' means : need a single number which we can use as a measure of the 'size' of $x_e(t)$ across the whole interval.
- Might appear sensible to take the error to be the average value of the $x_e(t)$ signal over $t_1 < t < t_2$. But then +ve and -ve errors tend to cancel

out.

- Better to take the error as the average (mean) square value of $x_e(t)$. (If $x_e(t)$ was voltage this would be the same as minimising the error power (or also minimising the rms error.)).

2.2 Minimising the squared error

Consider the Mean Squared Error $E(C_{12})$ as a function of C_{12} i.e. we want to see what the mean squared error between the original signal and the approximated signal is, as we change C_{12}.

$$\begin{aligned}E(C_{12}) &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} x_e^2(t)dt \\ &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(x_1(t)-C_{12}x_2(t)\Big)^2 dt\end{aligned}$$

To get value of C_{12} which minimises this function, set $d/dC_{12} = 0$ and solve for C_{12}

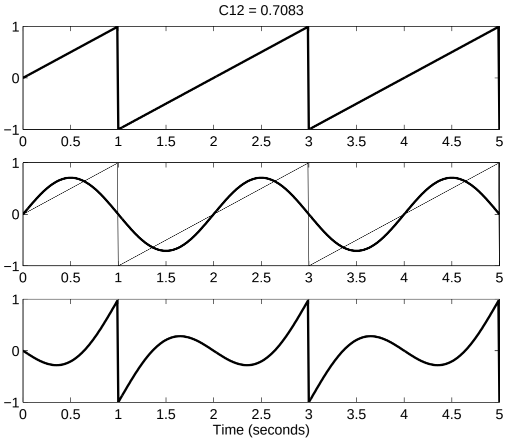
$$\begin{aligned}\frac{dE(C_{12})}{dC_{12}} &= \frac{d}{dC_{12}}\left[\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(x_1(t)-C_{12}x_2(t)\Big)^2 dt\right] \\ &= \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}}x_1^2(t)dt - 2\int_{t_1}^{t_2}\frac{d}{dC_{12}}x_1(t)C_{12}x_2(t)dt \right.\\ &\left.+ \int_{t_1}^{t_2}\frac{d}{dC_{12}}C_{12}^2x_2^2(t)dt\right]\end{aligned}$$

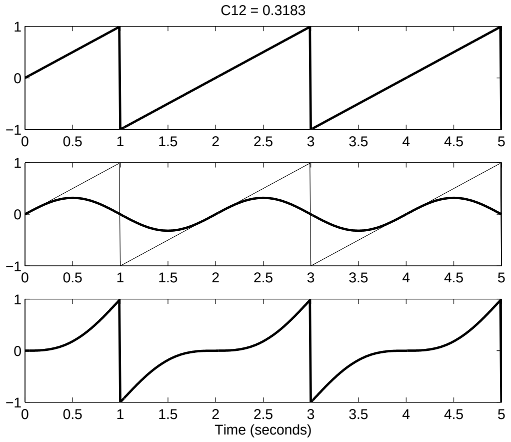
Hence, setting differential to zero gives

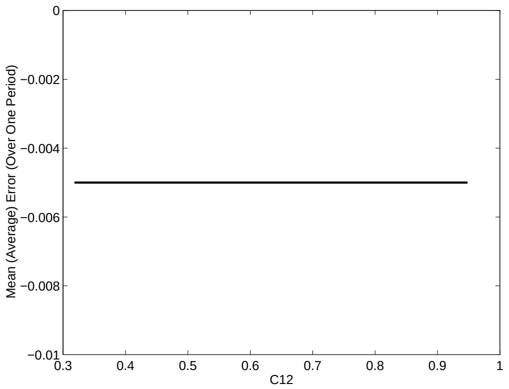
$$\frac{1}{t_2-t_1}\left[-2\int_{t_1}^{t_2} x_1(t)x_2(t)dt + 2C_{12}\int_{t_1}^{t_2} x_2^2(t)dt\right] = 0$$

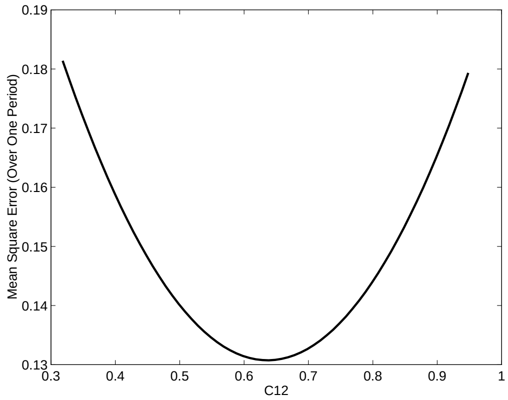
So:

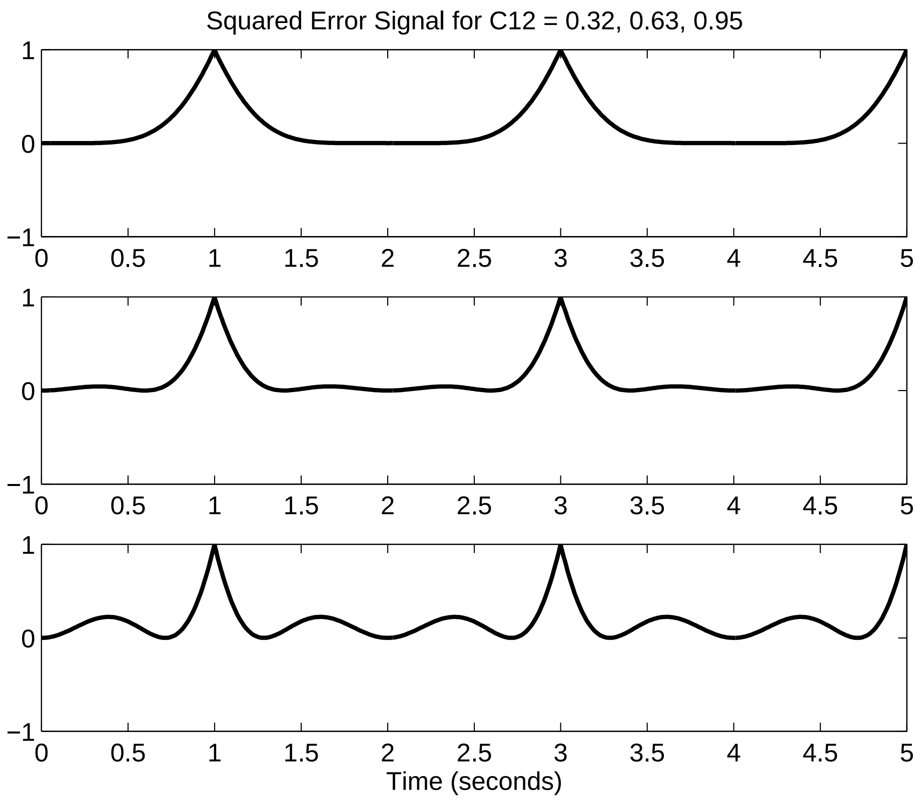
$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)x_2(t)dt}{\int_{t_1}^{t_2} x_2^2(t)dt}$$


C12 = 0.7083
Time (seconds)


C12 = 0.3183
Time (seconds)


Mean (Average) Error (Over One Period)
C12


Mean Square Error (Over One Period)
C12


Squared Error Signal for C12 = 0.32, 0.63, 0.95
1
0
−1
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5
Time (seconds)

Orthogonality continued

- By direct analogy with the vector argument; if C_{12} is zero we say that $x_1(t)$ contains no component of $x_2(t)$ and so the two signals are *orthogonal* in the interval $t_1 < t < t_2$.
- Therefore, if $\int_{t_1}^{t_2} x_1(t)x_2(t)dt = 0$ then $x_1(t)$ and $x_2(t)$ are orthogonal.
- Conversely, if $x_1(t) = x_2(t)$ over the selected interval then C_{12} must equal unity.
- Consider approximating the sawtooth by a single sinusoid at the fundamental frequency ω_0, ignoring any second and third order harmonics. Since both the sawtooth and the fundamental are strictly periodic over one period T_0, any approximation over one period *must* be valid for all other periods of the waveform; hence for all time. So we need only calculate C_{12} to approximate the sawtooth over one period only.

  The sawtooth is defined by

$$x_1(t) = \begin{cases} \frac{2t}{T_0} \text{ For } 0 \leq t \leq T_0/2 \\ \frac{2t}{T_0} - 2 \text{ For } T_0/2 \leq t \leq T_0 \end{cases} \quad (2)$$

  (Remember we can write this all in terms of ω_0 as well since $T_0 = 2\pi/\omega_0$.)

  And we wish to approximate this over the interval $0 < t < 2\pi/\omega_0$ $(0 < t < T_0)$ by

$$x_2(t) = C_{12} \sin \omega_0 t$$

- Simply substitute in the previous expression for C_{12} to give $C_{12} = 2/\pi$!

$$\int_0^{T_0} x_1(t)x_2(t)dt = 0 \tag{3}$$

- Therefore the *amount* of $x_2(t) = \sin(\omega_0 t)$ present in the sawtooth is $(2/\pi)\sin(\omega_0 t)$. **Any** other amount would give a larger *mean square error* over a complete period.

- It is interesting that the amplitude of the fundamental component in the Fourier series for the sawtooth is indeed the same value of $2/\pi$. This is because the Fourier method for deriving the 'amounts' of each sinusoid present is also based on a minimum mean square or 'least-square' error criterion.

- So .. big deal ... we can use least squares to show that the coefficients of the Fourier series expansion are selected to give the least-square error approximation to the actual signal. So what?

- This is 'what'

  – We can show that sines and cosines are also orthogonal over one period.

$$\int_0^{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)dt = 0$$
$$\int_0^{T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)dt = 0 \quad \text{for } n \neq m$$
$$\int_0^{T_0} \cos(n\omega_0 t)\sin(n\omega_0 t)dt = 0 \quad \text{for } n \neq m$$

  – Suppose we have approximated a periodic signal (e.g. sawtooth already considered) by its fundamental component. We now want to improve the approximation by adding in another harmonic. Its a pain if we then have to go through all the maths again, since now our approximating expression is different (two components instead of one). In other words, it is a nuisance if the incorporation of more components upsets the least-square error already achieved for the fundamental on its own. *BUT it may be shown that if the components are orthogonal to each other then recalculation is unnecessary. This is one valuable feature of Fourier analysis.*

So we can estimate 4 components, then add in a 5th without having to recalculate the 4 we just did.

- Sines and Cosines are not the only orthogonal basis set. There are several other such sets, including Legendre polynomials, Daubichies Wavelets, Haar functions etc. However, sine and cosines relate directly to our knowledge of system behaviour and human perception ... hence we study them and Fourier analysis is a powerful tool almost 200 years after it was proposed.

3 Fourier Series

The basic statement is

$$x(t) = A_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} C_k \sin(k\omega_0 t) \tag{4}$$

We can derive the expression for the coefficients A_0, B_k, C_k etc by substitution in the equation derived for C_{12} earlier, using $x_2(t) = 1, \cos(k\omega_0 t)$, $\sin(k\omega_0 t)$ respectively. Hence the coefficients are

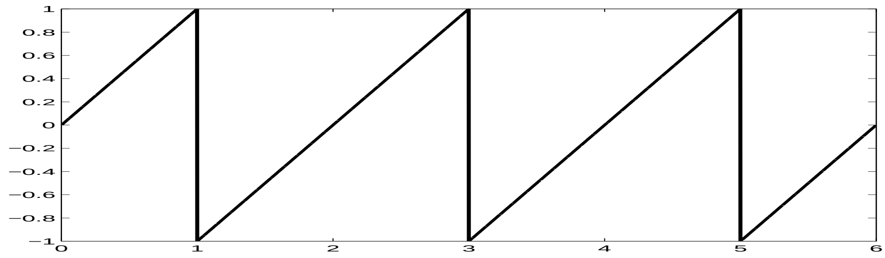
$$\begin{aligned}
A_0 &= \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} x(t)dt \quad \text{The average or DC value of the signal} \\
B_k &= \frac{\omega_0}{\pi} \int_0^{2\pi/\omega_0} x(t)\cos(k\omega_0 t)dt \\
C_k &= \frac{\omega_0}{\pi} \int_0^{2\pi/\omega_0} x(t)\sin(k\omega_0 t)dt
\end{aligned} \tag{5}$$

We can therefore find the ‘amount’ of any sine or cosine harmonic in a periodic signal $x(t)$ by multiplying the signal by that harmonic and integrating over one period.

NOTE THAT WE CAN USE ANY LENGTH OF ONE PERIOD.

ALSO NOTE THAT WE CAN DELETE THE SINE OR COSINE HARMONICS IF THE SIGNAL IS PURELY EVEN OR ODD RESPECTIVELY.

3.1 Sawtooth Wave example



Function is odd so only sine functions needed i.e. $B_k = 0$. $A_0 = 0$ because average value of signal (total area under curve) $= 0$

$$
\begin{aligned}
c_k &= \frac{\omega_0}{\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} x(t) \sin(k\omega_0 t) dt \\
&= \frac{\omega_0}{\pi} \int_{-1}^{1} t \sin(k\omega_0 t) dt
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
&= \frac{\omega_0}{\pi} \left[\frac{-t\cos(k\omega_0 t)}{k\omega_0} \right]_{-1}^{1} + \frac{\omega_0}{\pi} \int_{-1}^{1} \frac{\cos(\qquad)}{k\omega_0} \\
&= \frac{-2\cos(k\pi)}{k\pi} + \frac{\omega_0}{\pi} \left[\frac{\sin(k\omega_0 t)}{k^2\omega_0^2} \right]_{-1}^{1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
x(t) &= \sum_{k=1}^{\infty} \\
&= \frac{2}{\pi} \sin(\omega_0 t) - \qquad\qquad\qquad \ldots
\end{aligned}
$$

4 Complex form of the Fourier Series

It is possible to condense the form of the Fourier series expansion in equation 4 by employing complex exponentials. This *complex* Fourier series form is easier to manipulate since it is the same expression but uses fewer terms.

Using the identities

$$\sin(k\omega_0 t) = \frac{1}{2j}\left[e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right]$$
$$\cos(k\omega_0 t) = \frac{1}{2}\left[e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right]$$

The Fourier 'synthesis' equation

$$x(t) = A_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} C_k \sin(k\omega_0 t)$$

can then be written

$$x(t) = A_0 + \sum_{k=1}^{\infty} B_k \frac{1}{2}\left[e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right]$$
$$+ \sum_{k=1}^{\infty} C_k$$

And we can collect the similar exponential terms together to yield

$$x(t) = A_0 + \sum_{k=1}^{\infty} \frac{1}{2}\left[B_k + C_k/j\right] e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \frac{1}{2}\left[B_k - C_k/j\right] e^{-jk\omega_0 t}$$
$$= A_0 + \sum_{k=1}^{\infty} \frac{1}{2}\left[B_k - jC_k\right] e^{jk\omega_0 t} + \sum_{k=1}^{\infty} \frac{1}{2}\left[B_k + \qquad\right] e^{-jk\omega_0 t}$$

Hence

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{\infty} e^{jk\omega_0 t}\alpha_k + \sum_{k=1}^{\infty} e^{-jk\omega_0 t}\alpha_k^* \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \end{aligned} \tag{6}$$

So

$$a_k = \begin{cases} A_0 & \text{for } k = 0 \\ \frac{B_k - jC_k}{2} & \text{For } k > 0 \\ \frac{B_k + jC_k}{2} & \text{For } k < 0 \end{cases} \tag{7}$$

And from the expressions for B_k, C_k given in equation 5, we can derive a_k explicitly in terms of $x(t)$ by substituting as follows

$$\begin{aligned} a_k &= \frac{\omega_0}{2\pi}\left\{\int_{-\pi/\omega_0}^{\pi/\omega_0} x(t)\cos(k\omega_0 t)dt - j\int_{-\pi/\omega_0}^{\pi/\omega_0} x(t)\sin(k\omega_0 t)dt\right\} \\ &= \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} x(t)\Big[\cos(k\omega_0 t) - j\sin(k\omega_0 t)\Big]dt \\ &= \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} x(t)e^{-jk\omega_0 t}dt \end{aligned}$$

REMEMBER $\omega_0 = \frac{2\pi}{T_0}$

5 FOURIER SERIES: FINAL EXPRESSIONS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} x(t) e^{-jk\omega_0 t} dt \qquad (8)$$

OR

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T_0}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-2\pi jkt/T_0} dt \qquad (9)$$

Where $x(t)$ is **periodic**, T_0 is the **period** of $x(t)$ and $\omega_0 = 2\pi/T_0$. T_0 has units of SECONDS and ω_0 has units of RADIANS PER SECOND

- The concept of *negative* frequency has been introduced as a natural extension of the 'real' form of the Fourier series.
- Note that in plotting line spectra using this version of the series, the spectrum now posesses some symmetry about $\omega = 0$ and also the size of the frequency components is less than what they were for the real form of the series.
- Using this complex series we can now talk about the 'two-sided' bandwidth of a signal; whereas with the real series we would talk about a 'one-sided' badwidth.

5.1 Pulse waveform

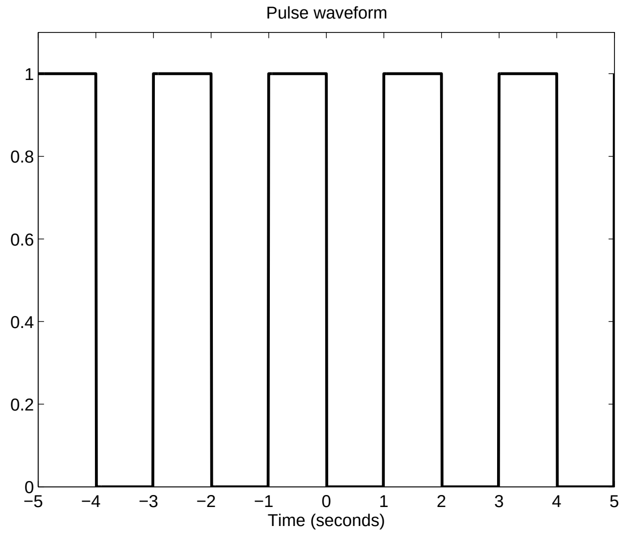
One kind of digital information signal (each pulse is a binary **1**).

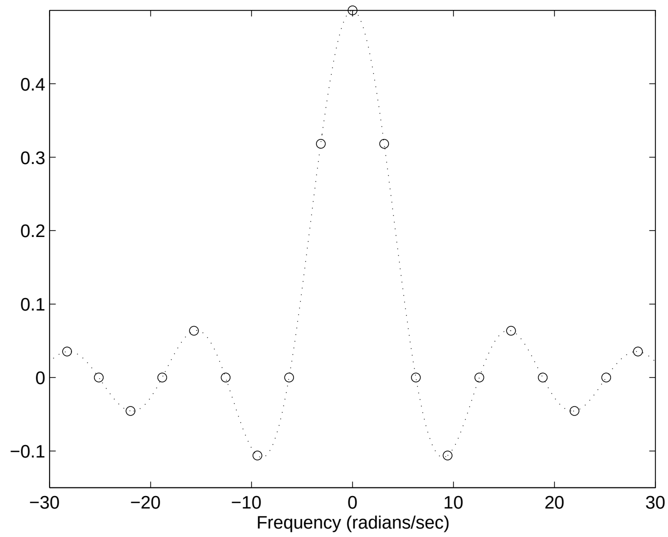
$$
\begin{aligned}
x(t) &= \sum_{k=\infty}^{\infty} a_k e^{-j2\pi kt/T_0} \\
\text{Where} \quad a_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi kt/T_0} dt \\
&= \frac{1}{T_0}\int_{-T_p/2}^{T_p/2} e^{-j2\pi kt/T_0} dt \\
&= \frac{1}{T_0}\left[\frac{-1}{j2\pi k/T_0} e^{\frac{-j\pi k T_p}{T_0}}\right]_{-T_p/2}^{T_p/2} \\
&= \frac{-1}{j2\pi k/T_0}\left[e^{\frac{-j\pi k T_p}{T_0}} - e^{\frac{j\pi k T_p}{T_0}}\right] \\
&= \frac{1}{\pi k}\frac{1}{2j}\left[e^{\frac{j\pi k T_p}{T_0}} - e^{\frac{-j\pi k T_p}{T_0}}\right] \\
&= \frac{1}{\pi k} \\
&= \frac{T_p}{T_0}\frac{\sin\left(\frac{\pi k T_p}{T_0}\right)}{\frac{\pi k T_p}{T_0}} \\
&= \frac{T_p}{T_0}\text{sinc}\left(\frac{\pi k T_p}{T_0}\right)
\end{aligned}
$$

Hence

$$
x(t) = \sum_{k=\infty}^{\infty} \underbrace{\left[\frac{T_p}{T_0} \qquad\qquad\qquad\right]} \overbrace{e^{-j\frac{2\pi kt}{T_0}}}
$$

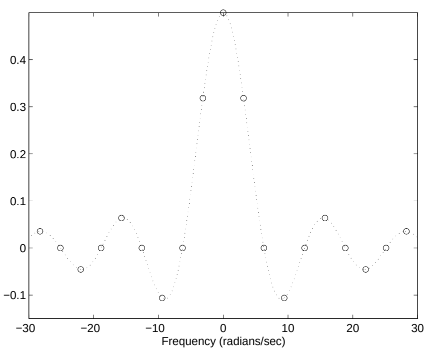
The Spectrum of the pulse train

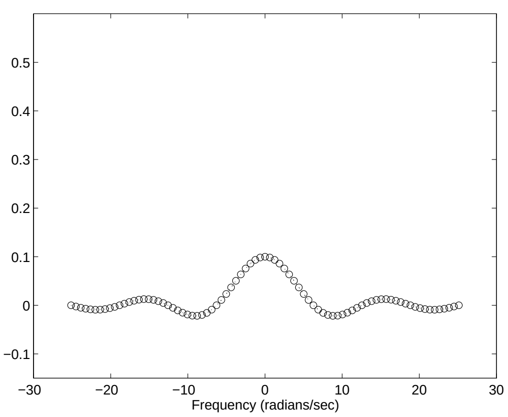




6 The Spectrum of a-periodic signals and the Fourier Transform

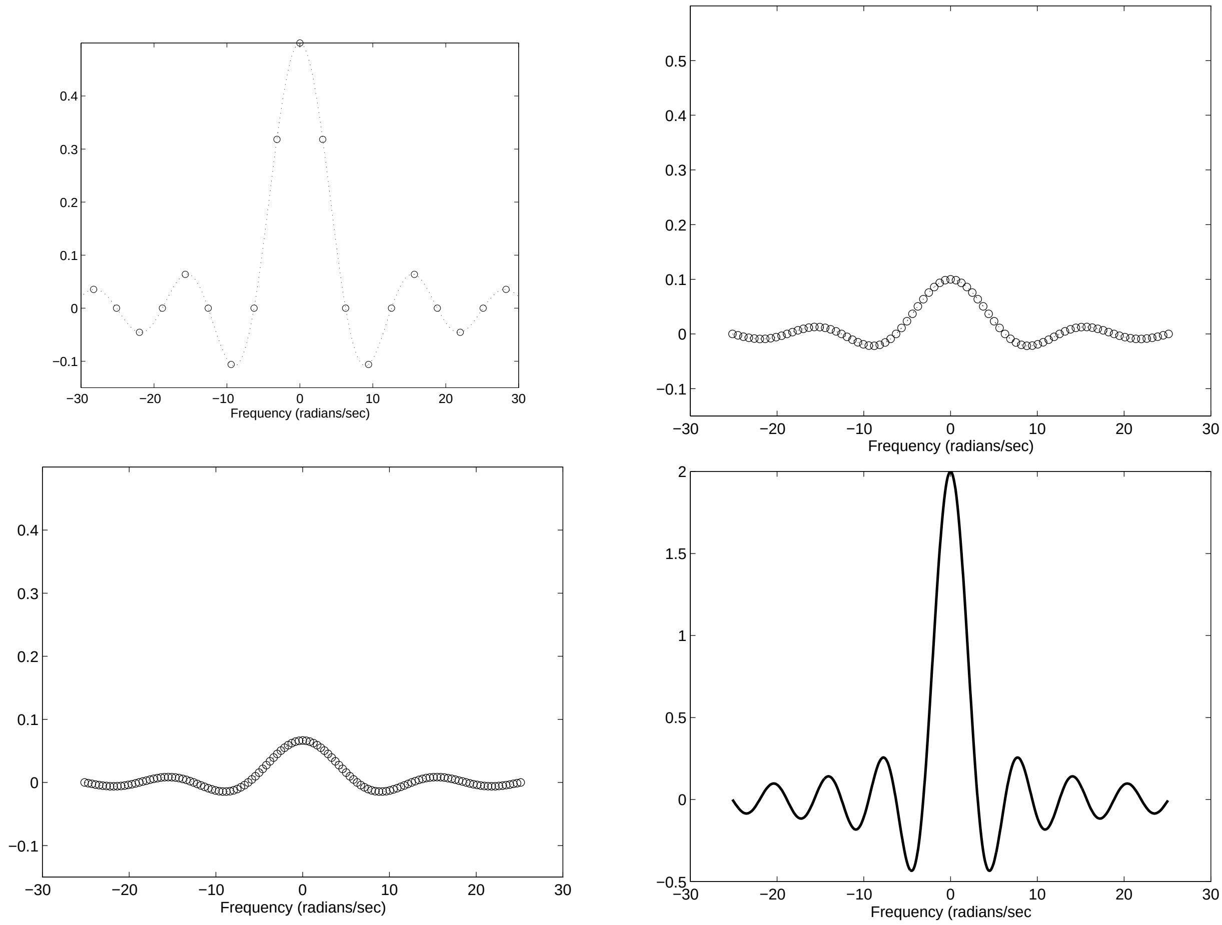
- The majority of interesting signals are not periodic. (speech, video).
- However, Fourier series provides a good starting point to introduce the notions of frequency spectra; and now we will investigate the spectrum of a-periodic signals and so introduce the FOURIER TRANSFORM
- Consider a single, isolated pulse. This can be manufactured by taking pulse train previously introduced, and letting $T_0 \to \infty$.
- In the limit, this pulse train will stop being periodic and become a single *time-limited* pulse.
- The components of the Fourier series expansion of the periodic signal become much more densley spaced as this happens, and the maximum amplitude of the line spectrum becomes very small.





6.1 The Fourier Transform (Hand waving explanations)

- We could have viewed the 'line spectra' (which we were previously drawing) as plots of a_k versus k.
- In the limit that the period of a signal tends to infinity, the Fourier series tends to the FOURIER TRANSFORM
- So 'discrete' frquency $k\omega_0$ becomes a continuous frequency ω; and the summation of the separate Fourier series components becomes an *integral* over a continuum of frequencies. The discrete coefficients a_k then become a continuous function of ω.



6.2 The Fourier Transform

Fourier series is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-2\pi jkt/T_0} dt$$

- As $T_o \to \infty$, a_k becomes very small. **But the product** $a_k T_0$ **does not vanish.** We choose to write this as a variable X.
- As $T_o \to \infty$, $\omega_0 \to 0$ and $k\omega_0$ tends to a continuous variable; denoted ω.
- Since X is a function of this new variable (continuous frequency) we will rewrite the second equation as

$$\begin{aligned} X(\omega) = a_k T_0 &= \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (10) \end{aligned}$$

- The first equation then becomes

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X(\omega)}{T_0} e^{jk\omega_0 t}$$

$$=$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt \qquad (11)$$

THE FOURIER TRANSFORM

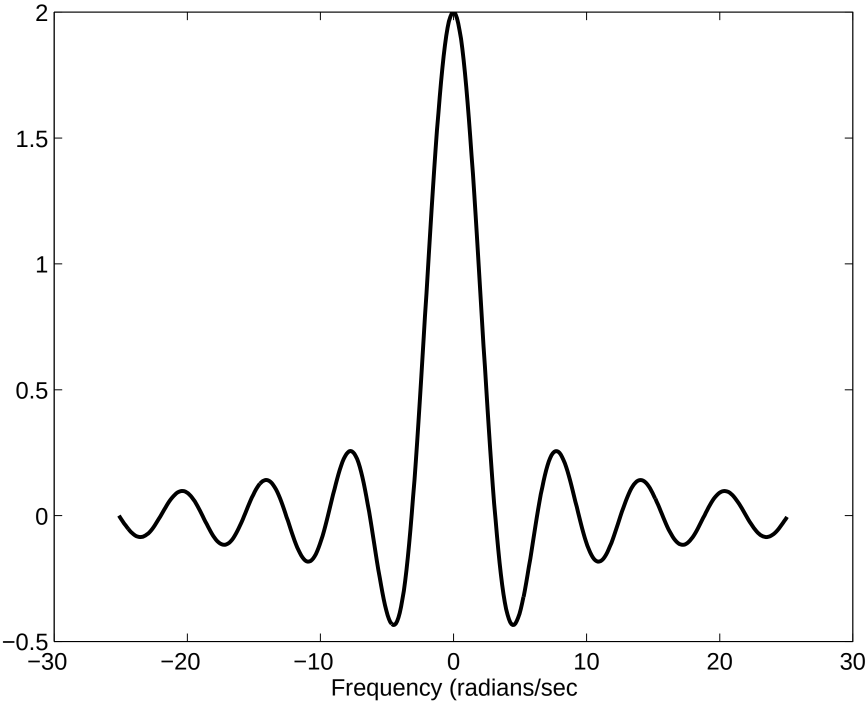
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt \qquad (12)$$

THIS IS IMPORTANT.

- In this continuous frequency domain, the component of $X(\omega)$ at any point-frequency is vanishingly small. We can no longer reliably refer to a single component in the way that we could do for the Fourier series.
- We talk instead about the energy contained over a small band of frequencies centred around that point. $X(\omega)$ is better thought of as a *frequency density* function.
- $\mathcal{F}\{x(t)\}$ denotes the Fourier transform of $x(t)$. $\mathcal{F}^{-1}\{X(\omega)\}$ deontes the inverse Fourier transform of $X(\omega)$.
- Fourier transform pair is denoted by :

6.3 FOURIER TRANSFORM OF A PULSE

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-T_p/2}^{T_p/2} e^{-j\omega t}dt \\
&= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T_p/2}^{T_p/2} \\
&= \frac{-1}{j\omega}\left[e^{-j\omega T_p/2} - e^{j\omega T_p/2}\right] \\
&= \frac{2}{\omega}\sin\left(\frac{\omega T_p}{2}\right) \\
&= T_p \\
&=
\end{aligned}$$



6.4 A relationship between the Fourier Transform and Laplace Transform

$$\text{The Laplace transform: } \mathbf{X}(s) = \int_0^\infty x(t)e^{-st}dt \tag{13}$$

$$\text{The Fourier Transform: } X(\omega) = \int_{-\infty}^\infty x(t)e^{-j\omega t}dt \tag{14}$$

- Remember that s is a complex number.
- For signals which are $= 0$ for $t < 0$. These expressions are the same putting $s = j\omega$.
- So the Fourier transform of a signal (which is 0 for $t < 0$) is the **same** as the Laplace transform of the signal with $s = j\omega$. In other words, it is the same as evaluating the Laplace transform surface along the *imaginary axis* only i.e. along $s = j\omega$.
- The Fourier transform is decomposing a signal in terms of a sum of sines and cosines i.e. weighted combinations of signals of the form $e^{j\omega t}$. The set of *basis* functions used are pure sinusoids.
- The Laplace transform generalises this to include damped sinusoids and decaying exponentials as the *basis* functions.
- Consider $s = \sigma + j\omega$ (say). Then e^{-st} is $e^{-\sigma t - j\omega t}$ Which is a decaying exponential $e^{-\sigma t}$ multiplied by a pure (complex) sinusoid $e^{-j\omega t}$.
- Most of the things we have done with the Laplace transform apply *directly* to the Fourier transform.

6.5 Convolution and the Fourier Transform

$$\mathcal{F}\{x(t) * h(t)\} = X(\omega) \times H(\omega)$$

Let $y(t) = x(t) * h(t)$. Then the proof is as follows:

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \Big[x(t) * h(t) \Big] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \Big] e^{-j\omega t} dt \end{aligned}$$

Swap the order of integration and note $x(\tau)$ does not depend on t

$$= \int_{-\infty}^{\infty} x(\tau) \Big[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \Big] d\tau$$

Changing the variable of integration to $u = t - \tau$

$$\begin{aligned} &= \int_{-\infty}^{\infty} x(\tau) \Big[\int_{-\infty}^{\infty} h(u) e^{-j\omega(u+\tau)} du \Big] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} \underbrace{\Big[\int_{-\infty}^{\infty} h(u) e^{-j\omega u} du \Big]} d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(\omega) d\tau \\ &= H(\omega) \int_{-\infty}^{\infty} \\ &= H(\omega) X(\omega) \end{aligned}$$

Fourier Transform properties

Linearity:

$$\mathcal{F}\{x_1(t) + x_2(t)\} = X_1(\omega) + X_2(\omega)$$

Shift in Time (used for motion estimation in some broadcast digital video products see www.snell.co.uk)

$$\begin{aligned} \text{If} \quad & x(t) \leftrightarrow X(\omega) \\ \text{Then} \quad & x(t-T) \leftrightarrow e^{-j\omega T} X(\omega) \end{aligned}$$

Shift in Frequency

$$\begin{aligned} \text{If} \quad & x(t) \leftrightarrow X(\omega) \\ \text{Then} \quad & X(\omega - \omega_0) \leftrightarrow e^{j\omega_0 t} x(t) \end{aligned}$$

Time warping

$$\begin{aligned} \text{If} \quad & x(t) \leftrightarrow X(\omega) \\ \text{Then} \quad & x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \end{aligned}$$

6.6 Fourier Transform properties : Proofs

Linearity:

$$\begin{aligned} F\left\{x_1(t)+x_2(t)\right\} &= \int_{-\infty}^{\infty}\left[x_1(t)+x_2(t)\right]e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty}x_1(t)e^{-j\omega t}dt + \int_{-\infty}^{\infty}x_1(t)e^{-j\omega t}dt \\ &= \end{aligned}$$

Shift In Time:

$$F\left\{x(t-\tau)\right\} = \int_{-\infty}^{\infty}x(t-\tau)e^{-j\omega t}dt$$

Change of variable:

$$\begin{aligned} &= \int_{-\infty}^{\infty}x(u)e^{-j\omega(u+T)}du \\ &= e^{-j\omega t}\int_{-\infty}^{\infty}x(u)e^{-j\omega u}du \\ &= \end{aligned}$$

Shift In Frequency:

$$\mathcal{F}^{-1}\Big[X(\omega-\omega_0)\Big] = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega-\omega_0)e^{j\omega t}d\omega$$

Change of variable

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(u)e^{j(u+\omega_0)t}du \\ &= \frac{1}{2\pi}\qquad\int_{-\infty}^{\infty}X(u)e^{j(ut)}du \\ &= \end{aligned}$$

Time Warping:

$$\mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

Change of variable

$$\begin{aligned} &= \frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-ju\omega/a}du \\ &= \frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-ju\frac{\omega}{a}}du \\ &= \frac{1}{a}X\left(\frac{\omega}{a}\right) \end{aligned}$$

For $a > 1 \Rightarrow$ Time contraction $\Rightarrow \frac{\omega}{a}$ Frequency dilate

For $0 < a < 1 \Rightarrow$ Time dilation $\Rightarrow \frac{\omega}{a}$ Frequency contract

6.7 Frequency Response

- We already know that the Laplace Transform of a system's impulse response is its transfer function.
- If we put $s = j\omega$ in the transfer function, this gives us the Frequency response of the system.
- But putting $s = j\omega$ is the same as taking the Fourier Transform
- **Thus the Fourier transform of the impulse response of a system is the system Frequency response.**

Proof:
Given impulse response $h(t) : t \geq 0$, we know that

$$\mathbf{H}(s) = \int_0^\infty h(t)e^{-st}dt$$

Frequency Response is had by putting $s = j\omega$ in Xfer function

$$\begin{aligned}\mathbf{H}(j\omega) &= \int_0^\infty h(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^\infty h(t)e^{-j\omega t}dt \quad \text{because } h(t) \text{ causal} \\ &= \mathcal{F}(h(t))\end{aligned}$$

Converseley, the impulse response of a system is the inverse Fourier Transform of the frequency response

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^\infty H(j\omega)e^{j\omega t}d\omega \qquad (15)$$

7 Fourier Xform of periodic signals

$x(t) = \delta(t) \Rightarrow X(\omega) = ?.$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= 1 \end{aligned}$$

$X(\omega) = \delta(\omega - \omega_0) \Rightarrow x(t) = ?.$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} d\omega \\ &= \frac{1}{2\pi} e^{j\omega_0 t} \end{aligned}$$

$x(t) = \sin(\omega_0 t) \Rightarrow X(\omega) = ?.$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \sin(\omega_0 t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2j} \Big[\qquad\qquad\qquad \Big] e^{-j\omega t} dt \end{aligned}$$

$x(t) = \cos(\omega_0 t) \Rightarrow X(\omega) = ?.$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2} \Big[\qquad\qquad\qquad \Big] e^{-j\omega t} dt \\ &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}$$

8 Overall observations

- a-periodic signals do not usually have line spectra. This is because you are trying to approximate a time limited signal with a bunch of signals (sines and cosines) that extend for all time. So you have to add alot of them together to get the resulting signal to be time limited as required.
- A signal with discontinuities has a wider bandwidth than those without
- Dilating a signal in time makes its bandwidth narrower. (Easy to remember: when you stretch a signal ..its lasting longer in time, hence you need less sines and cosines to represent it, since they are not time-limited.)
- Signals with wide bandwidth are 'narrower' in time than those with narrow bandwidths.

9 ENERGY

- If $x(t)$ is the voltage across a $R = 1\Omega$ resistance, then the instantaneous power is $x^2(t)/R = x^2(t)$. The total energy in $x(t)$ is thus:

$$\text{Energy} = \int_{-\infty}^{\infty} x^2(t)dt$$

- **In general**, we say that the energy of a signal $x(t)$ is given by

$$\text{Energy} = \int_{-\infty}^{\infty} x^2(t)dt$$

- Parseval's theorem relates the energy of a signal in time to the spectral density of the signal. It is

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- Parseval's relation shows that $|X(\omega)|^2$ has a physical interpretation as **energy density** (in Joules/Hertz) since the energy of $x(t)$ in the frequency range ω_0 to $\omega_0 + \delta\omega_0$ is $|X(\omega_0)|^2\delta\omega_0$.

$$\text{Energy Density Spectrum is } \; E(\omega) = |X(\omega)|^2$$

9.1 Parseval's Theorem: Proof

$$\int_{\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x(t)x(t)dt$$
$$= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right]dt$$

Swapping integrals

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\int_{-\infty}^{\infty}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X(-\omega)d\omega$$

But $X(-\omega) = X^*(\omega)$ for Real signals

$$\text{So} \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Hence Parseval says

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Since $X(\omega)$ is symmetric about the y-axis (even function)

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|^2 d\omega$$

9.2 POWER

For communications signals, the energy is usually infinite, so work instead with Power quantities.

We can find the average power dissipated by averaging over time

$$\text{Average power} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T^2(t)dt$$

where $x_T(t)$ is the same as $x(t)$ but truncated to zero outside the time window $-T/2$ to $T/2$.

Using Parseval we have:

$$\begin{aligned}
\text{Average power} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T^2(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega
\end{aligned}$$

We can define the **Power Spectral Density** (PSD) as:

$$\boxed{\mathcal{S}_x(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T}}$$

This has units of Watts/Hz.