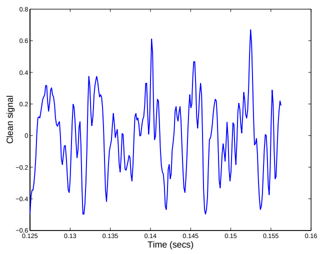
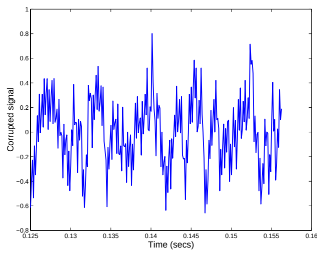
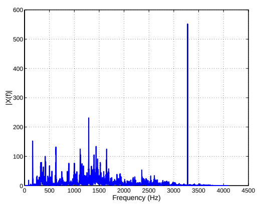
SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 6a
Dr Anil Kokaram
Electronic and Electrical Engineering Dept.
anil.kokaram@tcd.ie www.mee.tcd.ie\ ∼ack

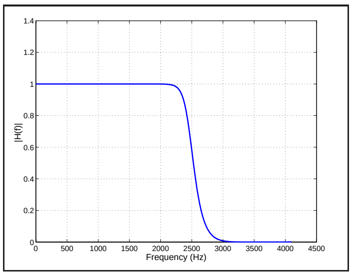
SIGNALS THROUGH FILTERS

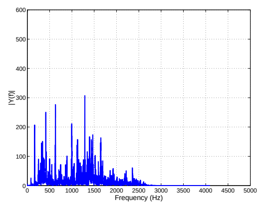
- Have just looked at Fourier analysis and already looked at the frequency response of signals
- This short handout gives an example of a signal interacting with ideal and real filters.

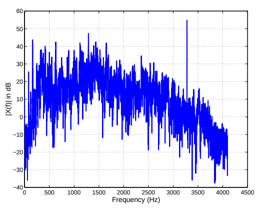


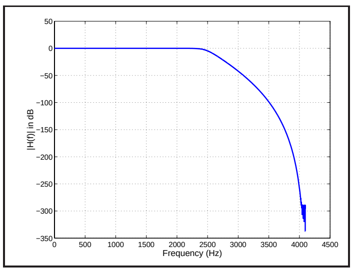


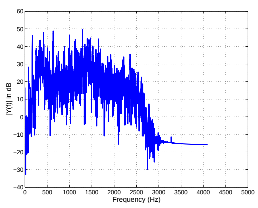












1 Ideal Filter

A signal $x(t)$ input into an ideal Band pass filter is given by $x(t) = 120e^{-24t}$ volts for $t \geq 0$ and 0 otherwise. The filter frequency response is ideal with a bandwidth of 4 rad/sec and an upper and lower cutoff of 48 and 24 rad/sec respectively. What percentage of the total energy of $x(t)$ is available at the output of the filter?

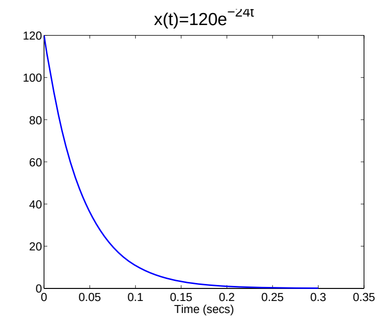
Let the signal at the output of the filter is $x_o(t)$

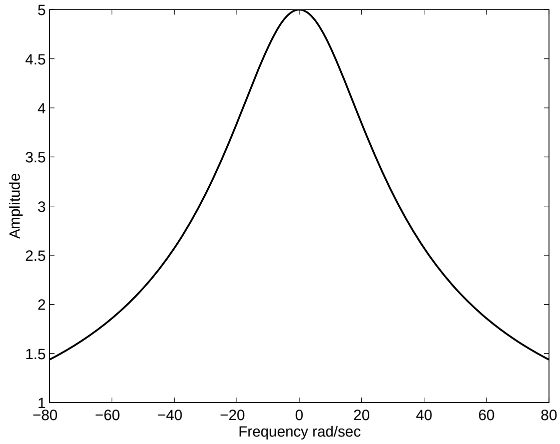
- First of all note that we can use Parseval's theorem to calculate the energy of the input signal by using the frequency domain.
- We need then to establish what the ideal band pass filter looks like in the frequency domain.
- This filter will cut off some components of frequency in the input signal $x(t)$ and the output is the sum of the remainder of the components. (That's why its called a filter).
- The output signal $x_o(t)$ can be calculated by finding the inverse transform of the altered frequency spectrum of the input signal i.e. $X(\omega)H(\omega)$ where $H(\omega)$ is the frequency response of the filter.
- But we are not interested in that, we want the energy of the output signal. This can again be obtained using Parseval's theorem since we are already in the frequency domain.

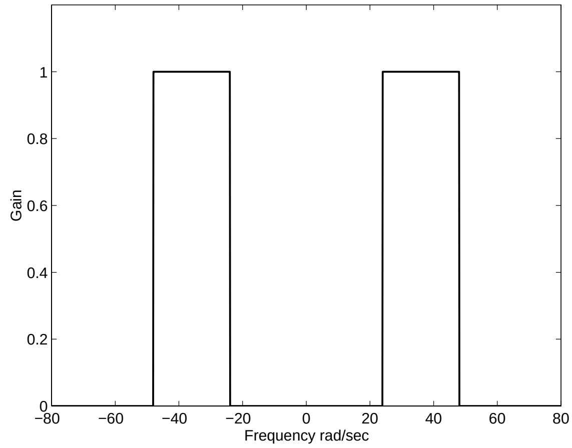
You'll need

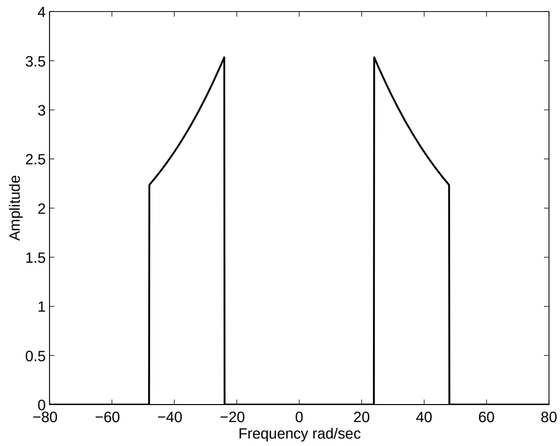
$$\int_0^\infty \frac{a}{a^2 + \omega^2} d\omega = \tan^{-1} \frac{\omega}{a}$$

1.1 Graphical Overview









Need to get frequency spectrum of input so ...
Fourier transform of $x(t) = 24e^{-120t}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_0^{\infty} \overbrace{120e^{-24t}}\, e^{-j\omega t}$$

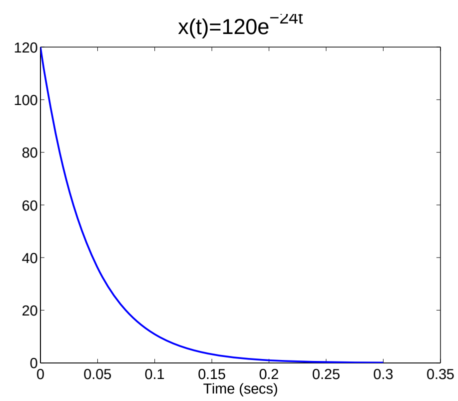
$$= 120 \int_0^{\infty} e^{-t(24+j\omega)}dt$$

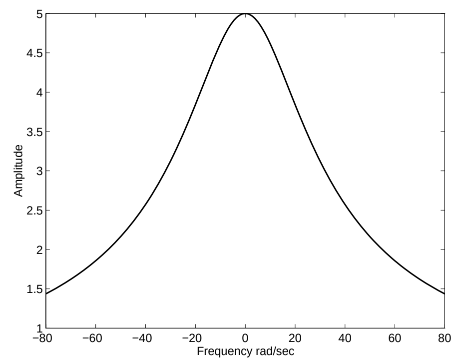
$$= 120 \left[\frac{e^{-t(24+j\omega)}}{-(24+\omega)} \right]_0^{\infty}$$

$$= \frac{120}{24 + j\omega}$$

Therefore

$$|X(j\omega)| = \frac{120}{\sqrt{24^2 + \omega^2}}$$





$$\text{Energy} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 d\omega$$

Using Parseval's theorem, the **energy at the input** to the filter is

$$W_i = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \qquad\qquad d\omega$$

$$= \frac{14400}{24\pi}\int_0^{\infty} \frac{24}{24^2+\omega^2} d\omega$$

$$= \frac{14400}{24\pi}\Big[\qquad\qquad \Big]_0^{\infty}$$

$$= \frac{14400}{24\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{14400}{24\pi}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{600}{\pi} \times \frac{\pi}{2}$$

$$= 300 \text{ Joules}$$

At the output of this *ideal* filter, the signal has all its frequency components **outside** of the pass band of the filter set to 0, otherwise the components are the same as the input. This means that everything **inside** $24 \leq 48$ rad/sec is left alone ie $X_o(\omega) = X(\omega)$ for $24 \leq |\omega| \leq 48$.

So total energy at output W_o is

$$W_o = \frac{1}{\pi}\int_0^{\infty} |X_o(\omega)|^2 d\omega$$

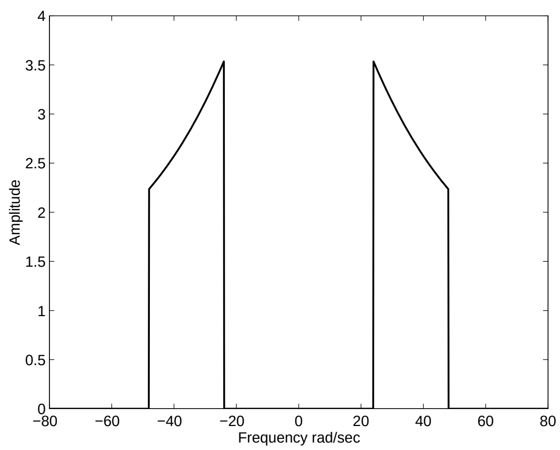
$$= \frac{1}{\pi}\int_{24}^{48} \frac{14400}{576+\omega^2} d\omega$$

$$= \frac{600}{pi}\left[\tan^{-1}\frac{10}{24}\right]_{24}^{48} \tag{1}$$

$$= \frac{600}{pi}\left[\tan^{-1} 2 - \tan^{-1} 1\right] \tag{2}$$

$$= 61.45\text{Joules}$$

So the percentage of the energy of $x(t)$ which appears in $x_o(t)$ is $\frac{W_o}{W_i} \times 100 = \frac{61.45}{300} \times 100 = 20.48\%$.



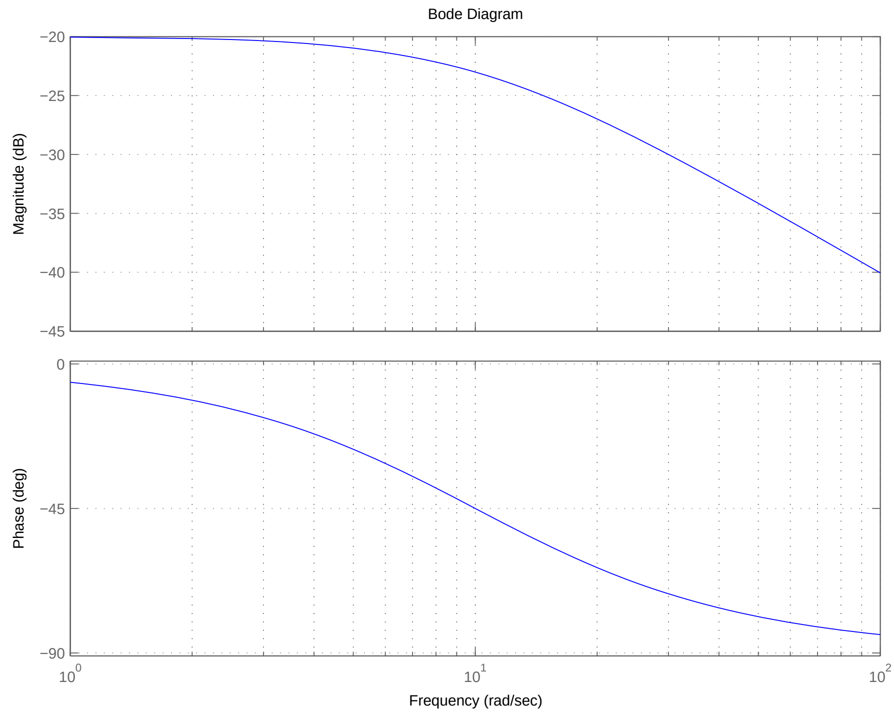
2 A Real (non-ideal) filter

Consider now, a filter whose response is not ideal. In other words, a 'real' filter. Given its frequency response is

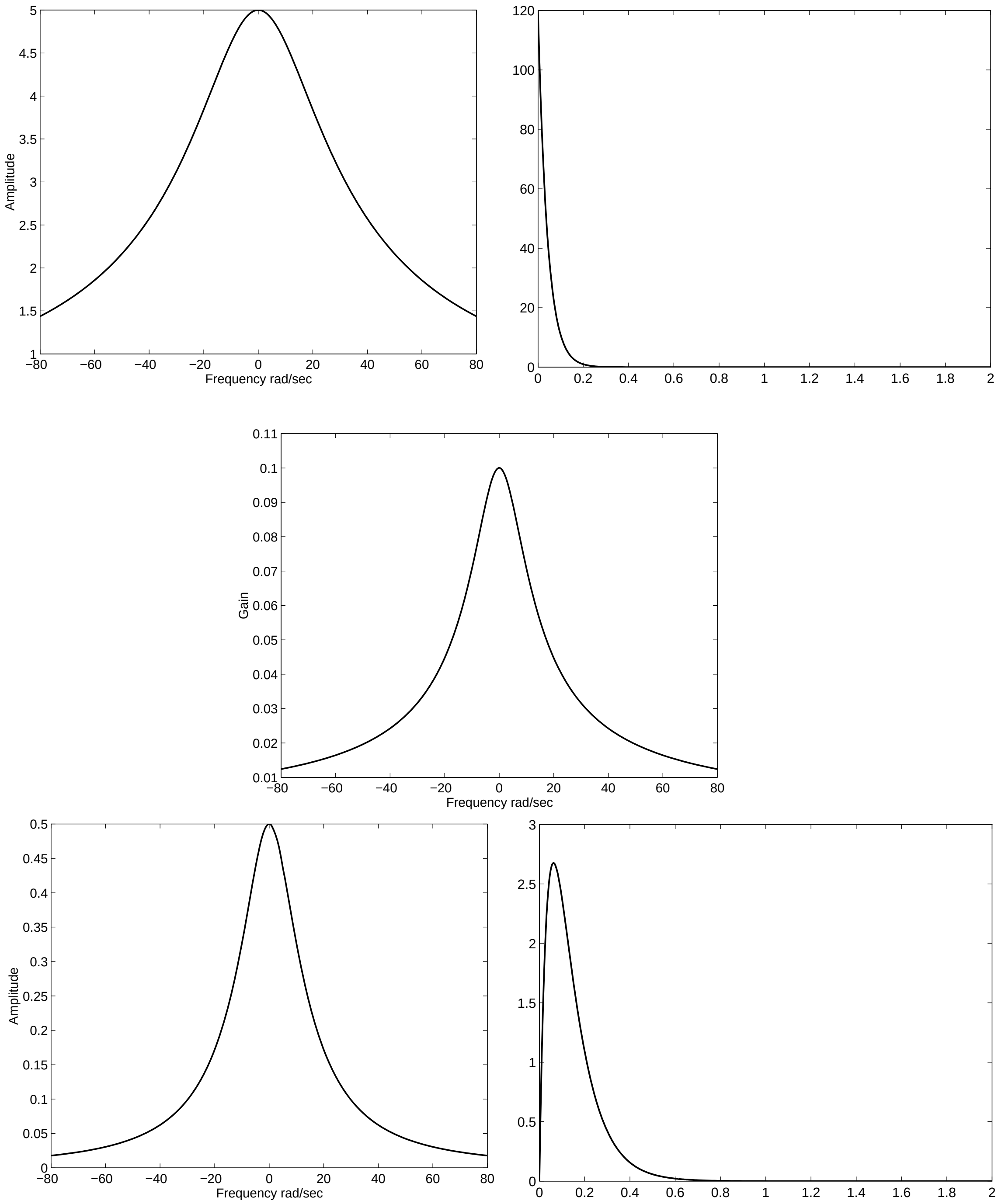
$$H(\omega) = \frac{1}{10 + j\omega}$$

what is the percentage of the energy of the input signal present at the output of this filter given the same $x(t)$ as before.

- This filter is *NOT* a bandpass filter. It is a low pass filter.
- The same ideas apply as before, but now it is a bit more tricky to calculate the frequency content of the output signal $x_o(t)$. Since the filter is not ideal, we actually have to do the system analysis to work out what the output is.



2.1 Graphical Overview



Thus

$$X_o(\omega) = X(\omega)\mathbf{H}(\omega) = \left(\frac{120}{24 + j\omega}\right)\left(\frac{1}{10 + j\omega}\right)$$

And now using Parseval, the energy of the output signal is

$$\begin{aligned}
W_o &= \frac{1}{\pi}\int_0^\infty |X_o(\omega)|^2 d\omega \\
&= \frac{1}{\pi}\int_0^\infty \frac{120^2}{|124 + j\omega|^2|10 + j\omega|^2} d\omega \\
&= \frac{1}{\pi}\int_0^\infty \frac{14400}{|24 + j\omega|^2|10 + j\omega|^2} d\omega \\
&= \frac{1}{\pi}\int_0^\infty \frac{14400}{(24^2 + \omega^2)(10^2 + \omega^2)} d\omega
\end{aligned}$$

A tricky integral : use partial fractions to break it up into something you know how to integrate.

$$= \frac{1}{\pi}\int_0^\infty \frac{A}{576 + \omega^2} d\omega + \frac{1}{\pi}\int_0^\infty \frac{B}{100 + \omega^2} d\omega$$

You know how to get the $A,\ B$ using the 'cover up' rule, hence

$$\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \frac{-30.25}{576 + \omega^2} d\omega + \frac{1}{\pi}\int_0^\infty \frac{30.25}{100 + \omega^2} d\omega \\
&= \frac{-30.25}{\pi}\left[\frac{1}{24}\tan^{-1}\frac{\omega}{24}\right]_0^\infty + \frac{30.25}{\pi}\left[\frac{1}{10}\tan^{-1}\frac{\omega}{10}\right]_0^\infty \\
&= 0.9 \text{ Joules}
\end{aligned}$$

So percentage of input energy that appears at output is $\frac{W_o}{W} \times 100 = \frac{0.9}{300} \times 100 = 0.3\%$.

Low pass filtering of images using a 2nd order Butterworth filter on rows then columns (separable 2-d filtering). The low pass filter coefficients were chosen to reduce the bandwidth of the image by a factor of 5.