SIGNALS AND SYSTEMS: PAPER 3C1
HANDOUT 8.

Dr Anil Kokaram
Electronic and Electrical Engineering Dept.
anil.kokaram@tcd.ie www.mee.tcd.ie/∼sigmedia

The Z Transform, System Transfer Function, Poles and Stability

- For differential equations and analogue system analysis, the Laplace Transform is an invaluable tool. It allows us to examine stability through the simple exercise of pole-zero plots and also allows us to understand frequency response of systems through factorisation.
- A similar tool exists for digital signals, it is called the Z Transform. It helps us avoid alot of tedious difference equation manipulation.
- You probably have had enough of transforms in this course, so rather than heap upon you a large amount of proofs, we will just be stating what the Z-Transform is and then we'll use it to do stuff.
- Where possible, we will be using similar analysis for Z- Transforms as we did for Laplace transforms and so the proofs and so on will be much more brief.

1 The Z-Xform

- The unilateral Z-Transform of a digital sequence x_n is given by

$$\mathcal{Z}(x_n) = \mathbf{X}(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

- z is just a complex number in what is called the $z-$plane. Just like s is a complex number in the $s-$ plane.
- The Z-Transform maps a discrete sequence x_n from the sample domain $[n]$ into the complex plane z.
- It is a power series in z

$$\boxed{\mathcal{Z}(x_n) = \mathbf{X}(z) = \sum_{n=0}^{\infty} x_n z^{-n}}$$

$$\mathcal{Z}(x_n) = \mathbf{X}(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

- Just to reassure you: the reaction of most undergraduates at this stage is "Wha?".
- An example. Say $x_n = 1,\ 2,\ -1,\ 0.5,\ 0.25,\ 0,\ 0$ and 0 otherwise, furthermore x_n is causal. The Z Transform of x_n is then as follows.

$$\begin{aligned}\mathbf{X}(z) &= \sum_{n=0}^{\infty} x_n z^{-n} \\ &= x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} \ldots \\ &= 1 + 2z^{-1} - 1z^{-2} + 0.5z^{-3} + 0.25z^{-4}\end{aligned}$$

And that's it. The Z-transform just takes an input sequence and multiplies it by increasing negative powers of z to create a power series in z.

- An important case. $x_n = r^n$ where $|r| < 1$

$$\mathbf{X}(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$
$$= \sum_{n=0}^{\infty} r^n z^{-n}$$

Sum to infinity of a Geometric Progression $a,\ ar,\ ar^2, ar^3, \ldots$ is $\frac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \left(rz^{-1}\right)^n$$

This is a GP with $a = 1$ and common ratio rz^{-1}. So

$$= \frac{1}{1 - rz^{-1}}$$

$$\boxed{r^n \leftrightarrow \frac{1}{1 - rz^{-1}}}$$

- What about the inverse? Well, the fancy way of doing the inverse is to do contour integration of $\mathbf{X}(z)$ in the z-plane to get back x_n. But we're going to use two simpler methods. One is just plain cheating and the other is to use tables[1].

[1] Also a form of cheating

1.1 The Inverse Z-Transform: The plain cheat

$$\mathcal{Z}(x_n) = \mathbf{X}(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

Suppose we are given $\mathbf{X}(z) = -1 + z^{-1} + 0.1z^{-2} + 0.8z^{-3} + 1.2z^{-4}$. What is x_n?
Well, the Z-Transform of x_n is a power series in z right?

$$\begin{aligned}\mathbf{X}(z) &= \sum_{n=0}^{\infty} x_n z^{-n} \\ &= x_0 z^0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} \dots\end{aligned}$$

Compare this with what we are given

$$\mathbf{X}(z) = -1 + z^{-1} + 0.1z^{-2} + 0.8z^{-3} + 1.2z^{-4}$$

You can see that the coefficients of the power series are the values of x_n themselves! So we can extract the inverse from the power series to yield

$$x_n = -1,\ 1,\ 0.1,\ 0.8,\ 1.2$$

Magic!

1.2 The Inverse Z-Transform: The tables cheat

Suppose we are given $\mathbf{X}(z)$ as follows. What is x_n?

$$\mathbf{X}(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} \tag{1}$$

Use partial fractions[2]

$$\frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}$$

Use cover up rule for $A,\ B$

$$\begin{aligned}
\text{Put } z^{-1} = 4 \text{ and cover up } \ A &= \frac{3 - \frac{20}{6}}{1 - \frac{4}{3}} \\
&= -1 \\
\text{Put } z^{-1} = 3 \text{ and cover up } \ B &= \frac{3 - \frac{15}{6}}{1 - \frac{3}{4}} \\
&= 2
\end{aligned}$$

Hence:

$$\begin{aligned}
\frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} &= \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}} \\
\mathcal{Z}^{-1}(\mathbf{X}(z)) = x_n &= -1\left(\frac{1}{4}\right)^n + 2\left(\frac{1}{3}\right)^n
\end{aligned}$$

[2] Yep, you can't escape them

2 Some Z Xforms

- Given $x_n = \delta_n$ what is $\mathcal{Z}(x_n)$?

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x_n z^{-n} \\ &= \sum_{n=0}^{\infty} \delta_n z^{-n} \end{aligned}$$

But δ_n is a sequence that is unity only where $n = 0$, otherwise its 0. Hence

$$\begin{aligned} &= 1 \times z^{-0} + 0 \times z^{-1} + 0 \times z^{-2} + \ldots \\ &= 1 \end{aligned}$$

- Given $x_n = u_n$ what is $\mathcal{Z}(x_n)$? (The Z-xform of a step function)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x_n z^{-n} \\ &= \sum_{n=0}^{\infty} u_n z^{-n} \end{aligned}$$

But u_n is a sequence that is unity for all $n \geq 0$. Hence

$$= \sum_{n=0}^{\infty} z^{-n}$$

Remember sum to ∞ of a GP is $a/(1-r)$, and here common ratio is z^{-1}

$$= \frac{1}{1 - z^{-1}}$$

3 Some Z-Xform Relations

- Convolution. Assuming x_n and e_n are causal what is the Z-Xform of the convolution of x_n and e_n?

$$\text{Let} \quad y_n = x_n * e_n$$

$$y_n = x_n * e_n = \sum_{k=0}^{\infty} x_k e_{n-k}$$

$$\Rightarrow \mathcal{Z}\{y_n\} = \sum_{n=0}^{\infty} y_n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} x_k e_{n-k} \right) z^{-n}$$

Let $m = n - k$

$$= \sum_{m=-k}^{\infty} \left(\sum_{k=0}^{\infty} x_k e_m \right) z^{-(m+k)}$$

Collect together the summations of like terms . . .

$$= \left(\sum_{k=0}^{\infty} x_k z^{-k} \right) \left(\sum_{m=-k}^{\infty} e_m z^{-m} \right)$$

$$= \mathbf{X}(z)\mathbf{Y}(z)$$

This is THE SAME AS the relationship between TIME DOMAIN CONVOLUTION AND THE LAPLACE OR FOURIER XFORMS.

- Time shift (very important in digital systems theory). If $x_n \leftrightarrow \mathbf{X}(z)$ what is the Z-Xform of x_{n-1}?

$$\text{Let} \;\; y_n = x_{n-1}$$
$$\mathcal{Z}(y_n) = \sum_{n=0}^{\infty} y_n z^{-n}$$
$$= \sum_{n=0}^{\infty} x_{n-1} z^{-n}$$

Substitute $m = n - 1$

$$= \sum_{m=-1}^{\infty} x_m z^{-(m+1)}$$

But we are dealing with CAUSAL signals so ...

$$= \sum_{m=0}^{\infty} x_m z^{-(m+1)}$$
$$= \sum_{m=0}^{\infty} x_m z^{-m} z^{-1}$$
$$= z^{-1} \sum_{m=0}^{\infty} x_m z^{-m}$$
$$= z^{-1} \mathbf{X}(z)$$

So z^{-1} represents a shift in time of ONE SAMPLE. Hence $\mathcal{Z}\{x_n\} = z^{-1}\mathbf{X}(z)$; $\mathcal{Z}\{x_{n-2}\} = z^{-2}\mathbf{X}(z)$; $\mathcal{Z}\{x_{n-3}\} = z^{-3}\mathbf{X}(z)$; and so on.

- Note that we are deliberately ignoring initial conditions in these notes see tables for exact Z-Xforms. In this course your initial conditions are always going to be zero, so you can ignore it.

4 System Xfer Functions

- Now we are in a position to make it easier to manipulate difference equations using the Z-Xform, in the same way that we can use the Laplace Xform to help us solve differential equations.

- Lets look at our simple IIR difference equation example again

$$y_n - 0.9y_{n-1} = x_n \tag{2}$$

  Let's try to find an expression for the output $\mathbf{Y}(z)$ in terms of $\mathbf{X}(z)$. Take Z-Xforms of both sides remembering

$$\begin{aligned} y_n &\leftrightarrow \mathbf{Y}(z) \\ y_{n-1} &\leftrightarrow z^{-1}\mathbf{Y}(z) \\ x_n &\leftrightarrow \mathbf{X}(z) \end{aligned}$$

  So taking Z-Xforms we have

$$\begin{aligned} \mathbf{Y}(z) - 0.9\mathbf{Y}(z)z^{-1} &= \mathbf{X}(z) \\ \Rightarrow \mathbf{Y}(z)[1 - 0.9z^{-1}] &= X(z) \\ \Rightarrow \mathbf{Y}(z) &= X(z)\frac{1}{1 - 0.9z^{-1}} \\ \Rightarrow \frac{\mathbf{Y}(z)}{X(z)} &= \frac{1}{1 - 0.9z^{-1}} \end{aligned}$$

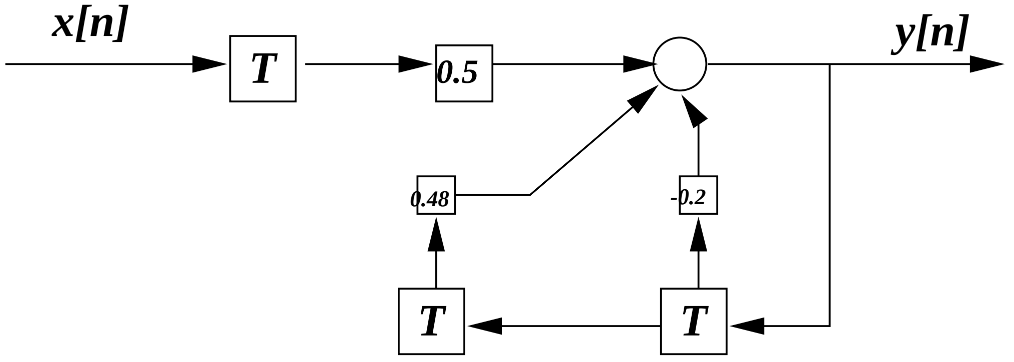
  The function $\frac{\mathbf{Y}(z)}{X(z)}$ is the SYSTEM TRANSFER FUNCTION for the difference equation. And given x_n we can use the above to work out the output given ANY input using the Z-Xform.

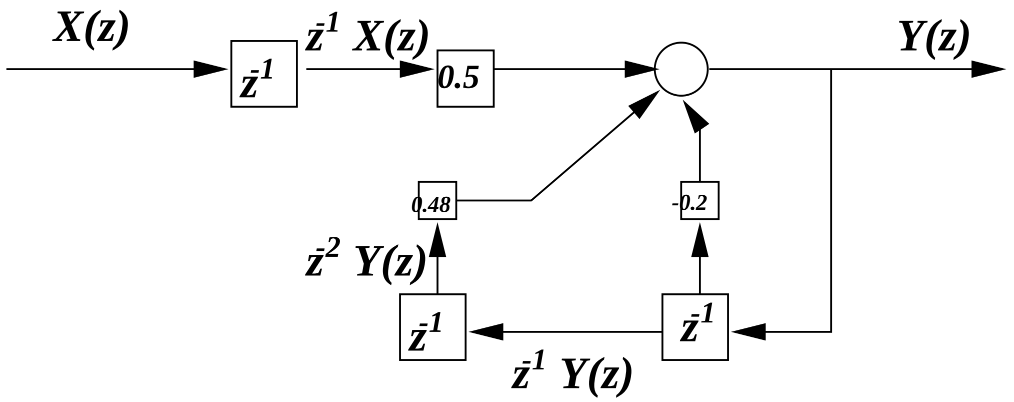
- ALL THE SAME METHODOLOGY AS FOR ANALOGUE SYSTEM TRANSFER FUNCTIONS ALSO APPLIES INCLUDING THE BLOCK DIAGRAM ALGEBRA.

- Thus if two systems $\mathbf{G}_1(z)$ and $\mathbf{G}_2(z)$ are in cascade the NET transfer function is $\mathbf{G}_1(z)\mathbf{G}_2(z)$.

4.1 BLOCK DIAGRAMS

$$y_n + 0.2y_{n-1} - 0.48y_{n-2} = x_n + 0.5x_{n-1}$$
$$\Rightarrow y_n = 0.48y_{n-2} - 0.2y_{n-1} + x_n + 0.5x_{n-1}$$
$$\mathbf{Y}(z) = \mathbf{Y}(z)[0.48z^{-2} - 0.2z^{-1}] + \mathbf{X}(z)[1 + 0.5z^{-1}]$$





4.2 Example 1

Given $x_n = \delta_n$ input into the system $G(z) = 1/(1 - 0.9z^{-1})$, what is the output h_n? (This is another way of asking you to calculate the impulse response of the system $G(z)$). Let h_n be the output sequence (the impulse response). BTW: $\mathbf{G}(z)$ is IIR.

$$
\begin{aligned}
\mathbf{H}(z) &= \mathbf{G}(z)\mathbf{X}(z) \\
\mathbf{X}(z) &= \mathcal{Z}(\delta_n) = 1 \\
\Rightarrow \mathbf{H}(z) &= \mathbf{G}(z) \\
\Rightarrow h_n &= \mathcal{Z}^{-1}(\mathbf{G}(z)) \\
&= \mathcal{Z}^{-1}(\frac{1}{1 - 0.9z^{-1}}) \\
\text{From tables} \quad &= 0.9^n \text{ Which we also calculated from first principles earlier!}
\end{aligned}
$$

THE SYSTEM IMPULSE RESPONSE IS THE INVERSE Z XFORM OF THE SYSTEM TRANSFER FUNCTION.

in exactly the same way that the time domain impulse response of an analogue system is the inverse Laplace Xform of its system transfer function.

4.3 Example 2

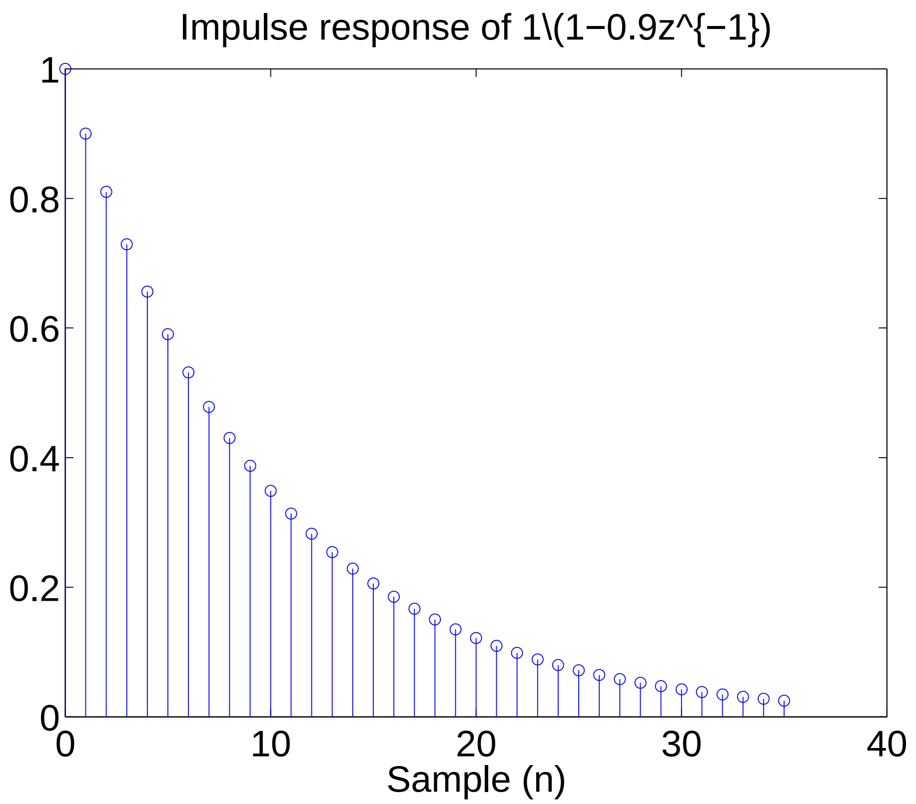
Given $x_n = u_n$ (the step function) input into the system $G(z) = 1/(1-0.9z^{-1})$, what is the output y_n? (This is another way of asking you to calculate the step response of the system $G(z)$). BTW: $\mathbf{G}(z)$ is IIR.

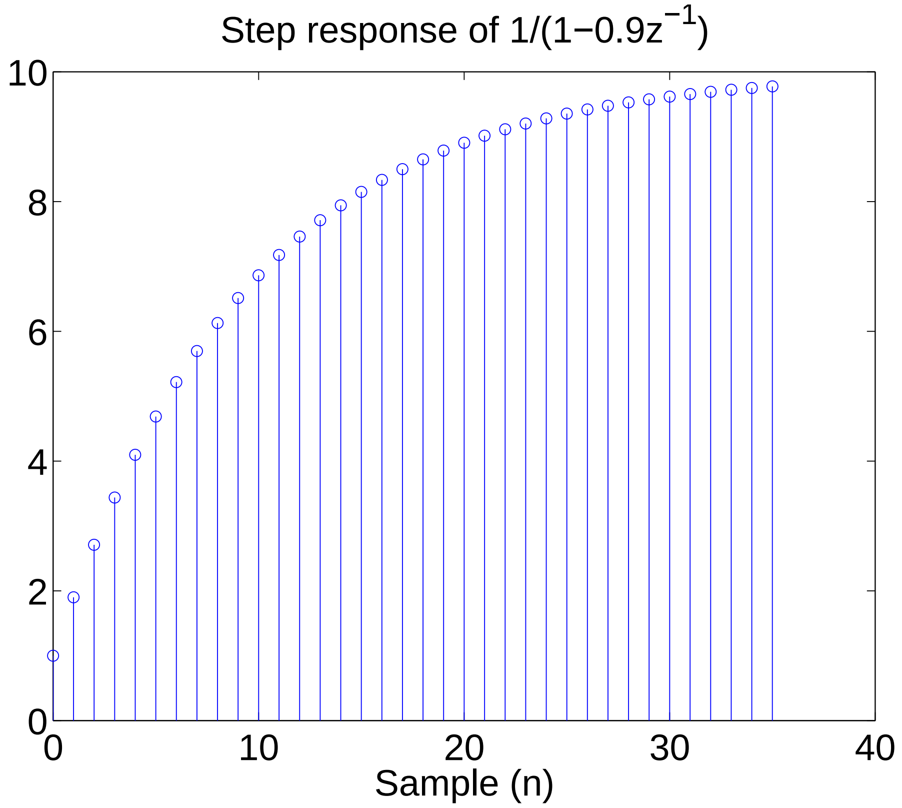
y_n is the output sequence (the step response in this case).

$$\mathbf{Y}(z) = \mathbf{G}(z)\mathbf{X}(z)$$
$$\mathbf{X}(z) = \mathcal{Z}(u_n) = \frac{1}{1-z^{-1}}$$
$$\Rightarrow \mathbf{Y}(z) = \mathbf{G}(z)\frac{1}{1-z^{-1}}$$
$$\Rightarrow y_n = \mathcal{Z}^{-1}\left(\frac{\mathbf{G}(z)}{1-z^{-1}}\right)$$
$$= \mathcal{Z}^{-1}\left(\frac{1}{(1-0.9z^{-1})(1-z^{-1})}\right)$$
$$\frac{1}{(1-0.9z^{-1})(1-z^{-1})} = \frac{-9}{1-0.9z^{-1}} + \frac{10}{1-z^{-1}}$$
$$y_n = -9(.9)^n + 10$$

THE SYSTEM STEP RESPONSE IS THE INVERSE Z XFORM OF [THE SYSTEM TRANSFER FUNCTION MULTIPLIED BY $\frac{1}{1-z^{-1}}$].

in similar fashion to analogue systems.

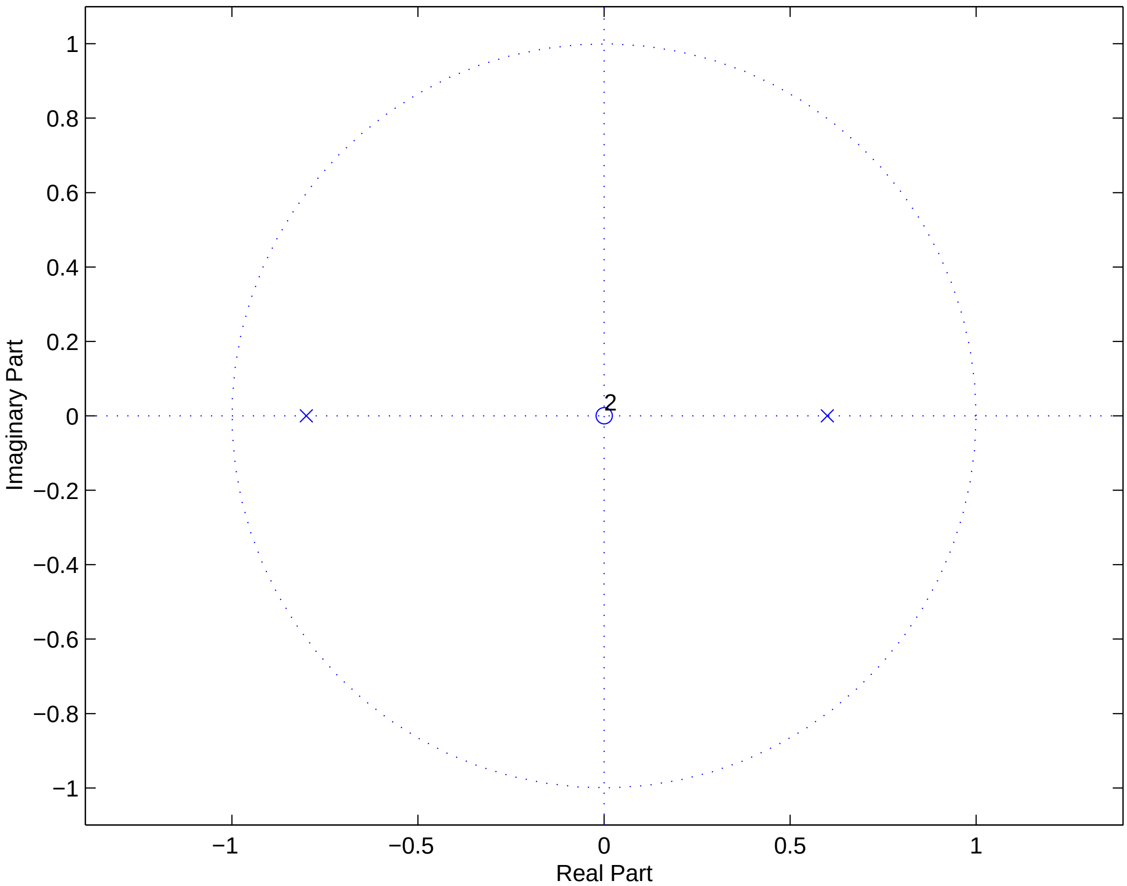

Impulse response of 1\(1−0.9z^{−1})
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
Sample (n)


Step response of 1/(1−0.9z−1)
10
8
6
4
2
0
0
10
20
30
40
Sample (n)

5 Poles and Zeros

- Same deal as for analogue systems. Poles are the values of **z** that make the denominator zero, and zeros are the values of **z** that make the numerator go to zero. NOT z^{-1}, just z !!!

$$\mathbf{G}(z) = \frac{1}{1 + 0.2z^{-1} - 0.48z^{-2}} = \frac{z^2}{z^2 + 0.2z - 0.48} = \frac{z^2}{(z+0.8)(z-0.6)} \quad (3)$$



You must always mark out the UNIT CIRCLE on the z-plane. We'll see why next ...

6 Stability

- Just like for analogue systems, a digital system is stable if its impulse response is absolutely summable i.e.

$$\sum_{n=0}^{\infty} |h_n| < 0$$

- Lets look at the system $\mathbf{G}(z)$ as below

$$\mathbf{G}(z) = \frac{1}{1 - az^{-1}}$$

  This system has a pole at $z = a$, and a zero at $z = 0$

- A DIGITAL SYSTEM IS STABLE IF ALL ITS POLES LIE WITHIN THE UNIT CIRCLE IN THE Z-PLANE.

- We will not prove this in depth. Instead we will use $\mathbf{G}(z)$ as an example.

- We know the impulse response of the system above is $h_n = 0.9^n$ Hence

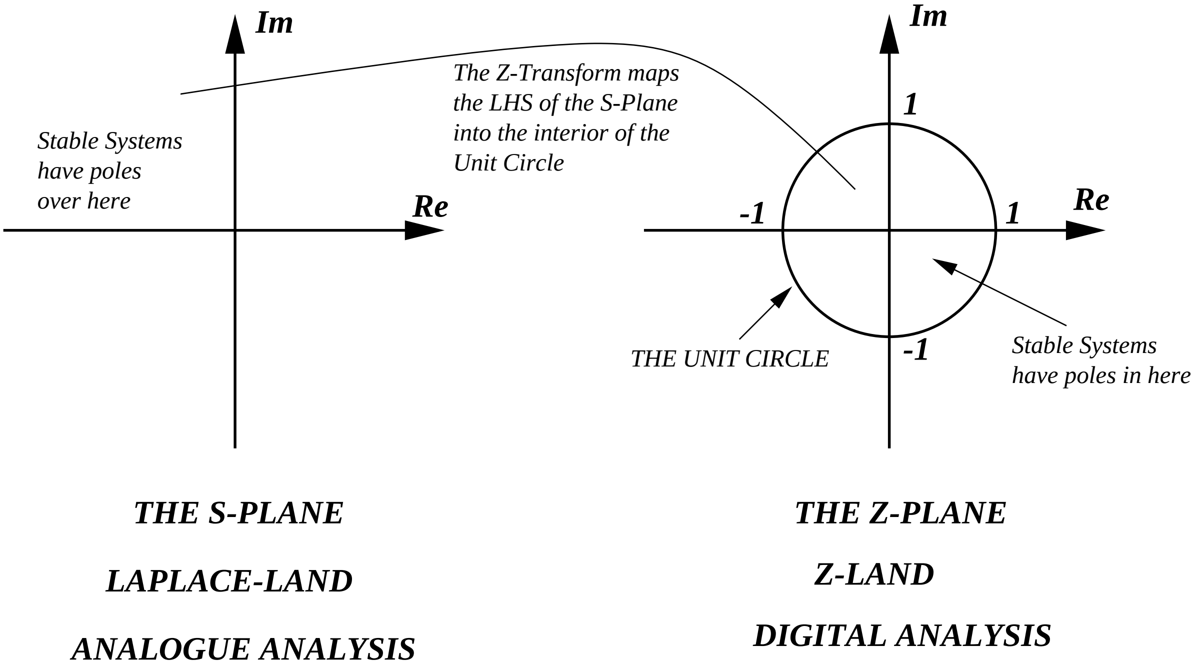
$$\sum_{n=0}^{\infty} |h_n| = \sum_{n=0}^{\infty} |a^n|$$

- For this sequence to be summable it must have a finite sum to ∞. Remember your GP..., the common ratio must therefore be < 1

- That means that $a < 1$ which means that the POLE must be within the unit circle!

- In general, to test stability for digital systems, just check that all the poles are within the unit circle. If any are not, then the system is unstable. We have not proved this exactly, but you can get the picture. The proof is almost identical to what we did using the Analogue System Transfer Functions.
- The location of zeros does not affect stability.
- **THEREFORE FIR FILTERS ARE ALWAYS STABLE!** They do not have poles!
- IIR Filters *always* have poles.
- If poles lie ON the unit circle then the system may be marginally stable. We will not deal with that in this course. All you will need to know is that z-plane poles must lie WITHIN the unit circle for a digital system to be stable.

A relationship between the s and z planes



We have not proved this, and most who see it for the first time figure its a stupid thing to notice anyway. I mean .. s isn't z right? So who cares if they are related?

It turns out that if you are designing digital systems that will be processing signals generated by sampling, and then the output will be reconstructed to give an analogue signal, this connection helps.

But we will not be dealing with that in this course. That's got more to do with design. In this course we just want you to get a handle on the tools, and the rough lay of the land.