

Root Locus

Signals and Systems: 3C1
Control Systems Handout 3

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- Recall, the example of the PI controller car cruise control system. A consequence of choosing the parameter for the integral control, K_i , is that it affects the location of the poles and zeros of the cruise control system on the s-plane. This different pole/zero locations for $K_i = 50$ and $K_i = 100$ are shown below

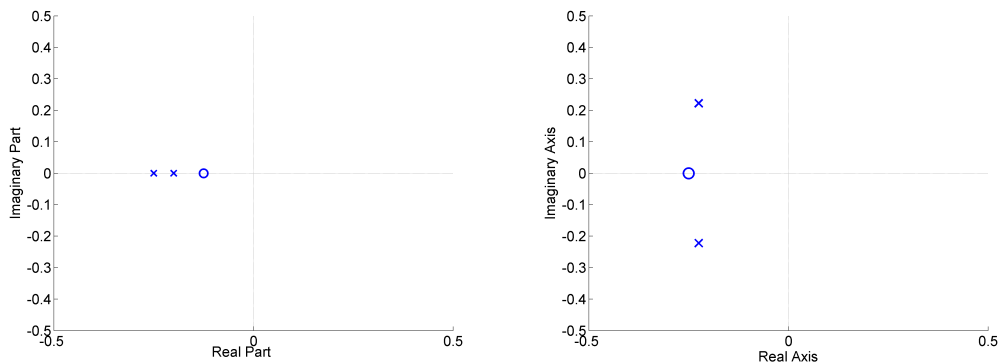


Figure 1: Pole/Zeros Diagrams in the car cruise control example for $K_i = 50$ and $K_i = 100$

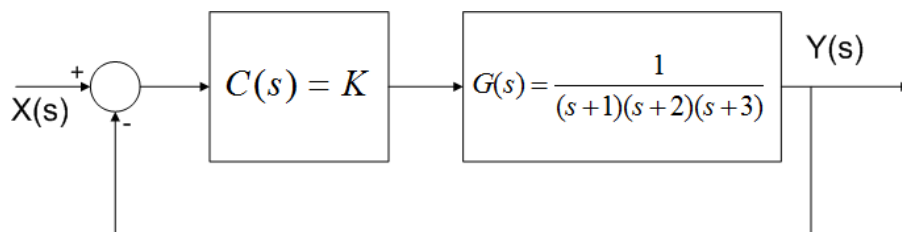
- Knowing the locations of the poles and zeros is important because
 1. It affects the transient response.
 2. It indicates whether the system is stable or not.

- An important property of the control system is the variation of these locations as the controller gain changes. Therefore we need to quantify these variations.
- In this handout we introduce the idea the **root locus** which as a graphical means of quantifying the variations in pole locations (but not the zeros).

1 Closed Loop System Stability

Recall that any system is stable if all the poles lie on the LHS of the s-plane. Ensuring stability for an open loop control system, where $H(s) = C(s)G(s)$, is straightforward as it is sufficient merely to use a controller such that the cascade $C(s)G(s)$ only has poles on the LHS of the complex plane. The poles of the open loop system will not be affected by the gain of the controller. *i.e.* If $C(s)G(s)$ is stable then $KC(s)G(s)$ will be stable for all values of K .

However for a closed loop system this will not be the case. Considering the following block diagram



we get a transfer function

$$\begin{aligned}
 H(s) &= \frac{C(s)G(s)}{1 + C(s)G(s)} \\
 &= \frac{K}{(s+1)(s+2)(s+3)} \\
 &= \frac{K}{1 + \frac{K}{(s+1)(s+2)(s+3)}} \\
 &= \frac{K}{s^3 + 6s^2 + 11s + 6 + K}
 \end{aligned} \tag{1}$$

Looking at the open loop transfer($H(s) = C(s)G(s)$) for this system, all the poles ($s = -1$, $s = -2$, $s = -3$) lie in the LHS of the s-plane and hence the system is stable.

For the closed-loop system if the gain $K = 10$ the poles lie in the LHS of the plane and so it is a stable system. However, if the gain is increased to $K = 200$ then there are 2 poles on the RHS of the s-plane and so the system is unstable.

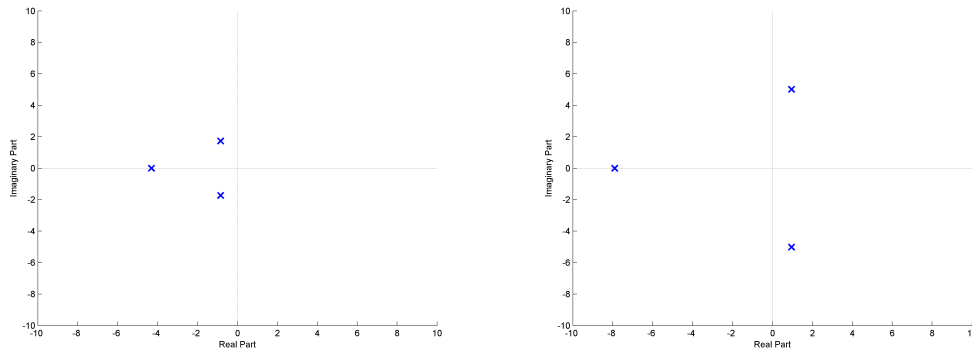


Figure 2: Pole Zero Plots for the system transfer function in Eq. (1) for $K = 10$ (left) and $K = 200$ (right).

It is also possible to have an unstable open loop system and a stable closed loop system. For example, consider the previous example where $G(s)$ is now

$$G(s) = \frac{s + 1}{(s - 1)(s + 2)(s + 3)}. \quad (2)$$

As $G(s)$ has a pole at $s = 1$, the open loop system $C(s)G(s)$ is clearly unstable. However in the closed loop system, if we choose $K = 10$ then the poles are located as shown in Fig. 3 and the system is stable.

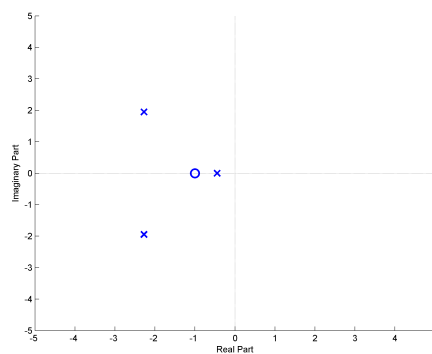
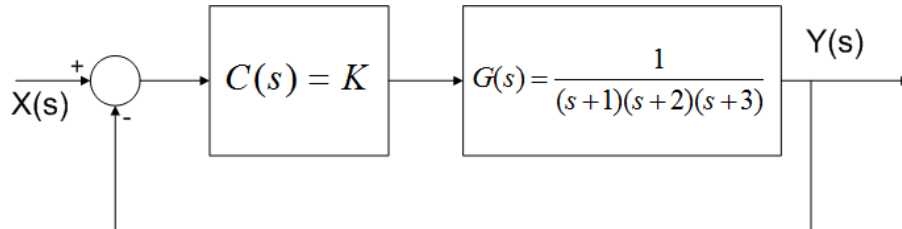


Figure 3: Pole Zero Plots for the system transfer function in Eq. (2) for $K = 10$.

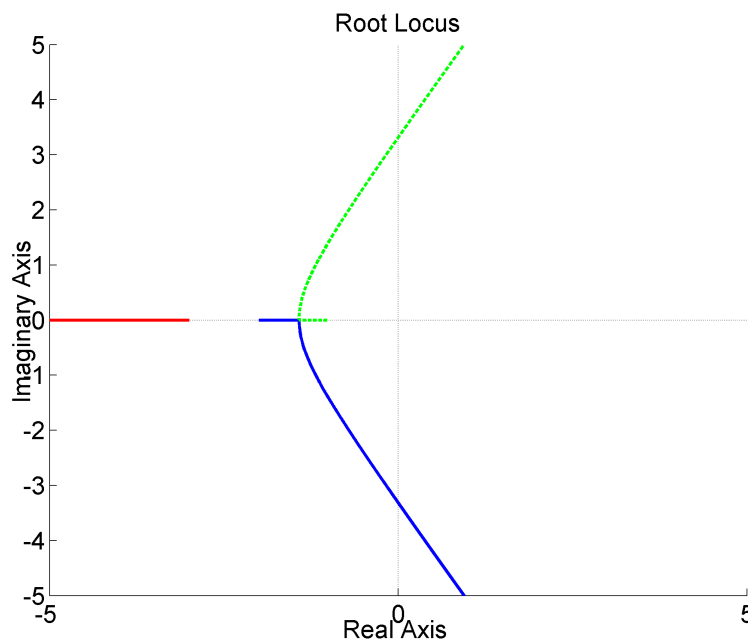
2 Root Locus



Revisiting the example in the last page, the system above has a transfer function

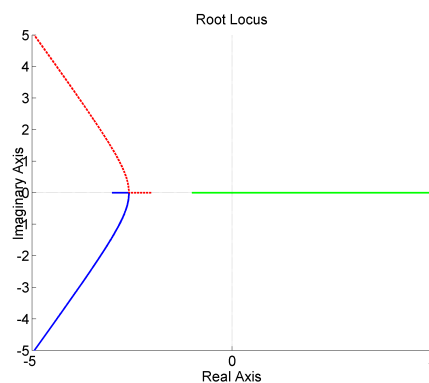
$$H(s) = \frac{K}{s^3 + 6s^2 + 11s + 6 + K}.$$

Starting at a value of $K = 0$ calculate the locations of the poles of $H(s)$ and repeat for increasingly large values of K . If we trace the locations of these poles as they move across the s -plane the following plot is obtained. This plot is called the root locus.



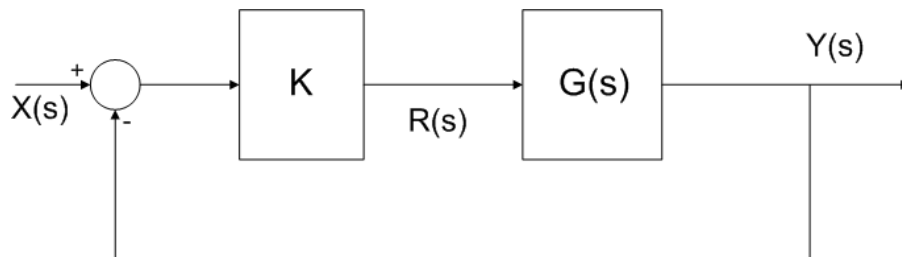
Observations

- Because we have a 3rd Order System, there are 3 separate plots on the root locus, one for each root.
- The plot is symmetric about the Real Axis. This is because complex roots occur in conjugate pairs.
- Each plot starts at a location equal to the location of a root of the plant transfer function. Each plot ends at infinity.
- The asymptotes of each plot as they tend infinity, share a common centre. Their angles are evenly spread out over 360°.
- Two of the plots cross the imaginary axis. This corresponds to the system becoming unstable with increasing K .
- All parts of the real axis to the left of an odd number of poles are part of one of the plots.
- 3 plots break off from the real axis when they meet. Before they break off the system has 3 real poles, after they break off 2 roots are complex.
- A root locus plot also exists for negative values of K . The root locus plot for $0 \geq K > -\infty$ shown below.



2.1 The Root Locus Concept

We will now look at the root locus in general and look in more detail at some of the observations we made from our example.



Consider a closed loop system above with unity feedback that uses simple proportional controller. It has a transfer function

$$H(s) = \frac{KG(s)}{1 + G(s)} = \frac{n(s)}{d(s)}$$

The poles occur at the roots of $d(s)$. For system transfer function above these roots occur where

$$1 + KG(s) = 0. \quad (3)$$

This is referred to as the **characteristic equation** of the system. Therefore it is necessary that

$$|KG(s)| = 1 \quad (4)$$

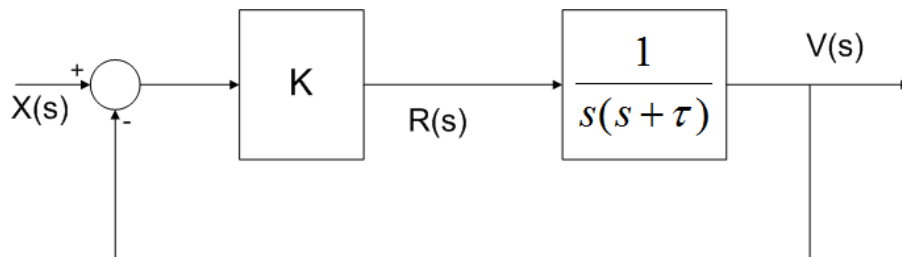
$$\angle G(s) = 180^\circ \pm k360^\circ \text{ for } k \in \mathcal{Z}. \quad (5)$$

These two equations are referred to as the **Magnitude** and **Angle** criteria respectively.

Therefore, the root locus is the path of the roots of the characteristic equation due to K as K is traced out to infinity.

Example

We will show that by manipulating the denominator polynomial it is possible to generate a root locus plot for the variation of other transfer function parameters.



For the system above the characteristic equation of the root locus due to variations in K can be written directly from Eq. 3 as

$$1 + K \frac{1}{s(s + \tau)} = 0 \text{ for a fixed } \tau.$$

However we can also rewrite the characteristic equation if we wished to see the effect of variations of τ on the roots of the control system

$$\begin{aligned} 1 + K \frac{1}{s(s + \tau s)} &= 0 \\ \Rightarrow s^2 + \tau s + K &= 0 \\ \Rightarrow 1 + \tau \frac{s}{s^2 + K} &= 0 \text{ for a fixed } K. \end{aligned}$$

Comment In general, it is often possible to generate a characteristic equation of the form in Eq. 3 for another choice of parameter. It can be also be used for more complicated closed loop system architectures. Although it is not necessary to generate the equation in this form to plot a root locus, it allows the keypoints of the root locus to be easily identified.

3 Root Locus Procedure

We will now show the procedure for rapid sketching of the root locus of a transfer function $H(s)$ as values of the tuning parameter K varies from $0 \leq K < \infty$. Before we start we note that as the complex poles of $H(s)$ will always occur in conjugate pairs the root locus will be symmetric about horizontal real axis.

Step 1: Prepare the sketch

Firstly, identify the parameter to be varied (K) and generate the appropriate characteristic equation of the form

$$1 + KP(s) = 0 \quad (6)$$

Secondly, we consider the positions of the roots of the characteristic equation for the start ($K = 0$) and end ($K = \infty$) points for the root locus. To do this, we factor $P(s)$ and write the polynomial in terms of its poles and zeros as follows

$$1 + K \frac{\prod_{i=1}^M (s - z_i)}{\prod_{j=1}^N (s - p_j)} = 0$$

where z_i and p_j are the zeros and poles of $P(s)$ respectively.

When $K = 0$, we can find the roots of the characteristic equation by re-writing it as

$$\begin{aligned} \prod_{j=1}^N (s - p_j) + K \prod_{i=1}^M (s - z_i) &= 0, \\ \Rightarrow \prod_{j=1}^N (s - p_j) &= 0. \end{aligned}$$

Hence when $K = 0$, the roots of the transfer function $H(s)$ are given by the the poles of $P(s)$.

When $K = \infty$, we can find the roots of the characteristic equation by re writing it as

$$\begin{aligned} \frac{1}{K} \prod_{j=1}^N (s - p_j) + \prod_{i=1}^M (s - z_i) &= 0, \\ \Rightarrow \prod_{i=1}^M (s - z_i) &= 0. \end{aligned}$$

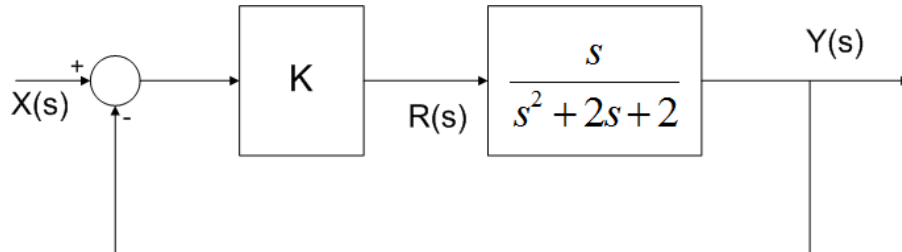
Hence when $K = \infty$, the roots of the transfer function $H(s)$ are given by the the zeros of $P(s)$.

Observations

- The number of separate plots on the root locus sketch will be the same as the number of poles of $P(s)$.
- Each plot will start at a pole of $P(s)$ for $K = 0$ and end at a zero of $P(s)$ for $K = \infty$.
- In general, system could have more poles than zeros. In this case, there will be a number of plots that tend to infinity, with the number being determined by the difference between the number of poles and zeros (*i.e.* . $N - M$). Such a system is said to contain $N - M$ zeros at infinity.

Case Study

Consider the closed loop system with a simple proportional controller as follows



The transfer function for this system is

$$H(s) = \frac{Ks}{s^2 + (2 + K)s + 2}.$$

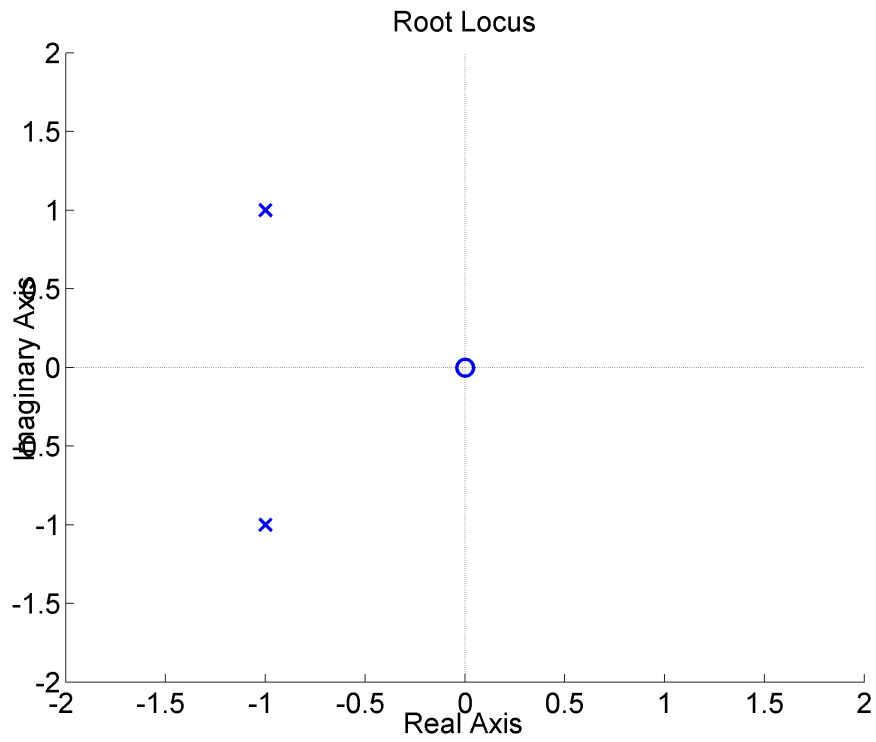
To get the root locus as K varies we obtain the characteristic equation from the denominator of $H(s)$ as follows

$$\begin{aligned} s^2 + (2 + K)s + 2 &= 0 \\ \Rightarrow s^2 + 2s + 2 + Ks &= 0 \\ \Rightarrow 1 + K \frac{s}{s^2 + 2s + 2} &= 0. \end{aligned}$$

We then find the zeros and poles of $P(s)$ which mark the start and end points of each branch. For

$$P(s) = \frac{s}{s^2 + 2s + 2}$$

we have one zero at $s = 0$ and two poles at $-1 \pm j$. These points are marked on the root locus plot



Step 2: Determine the Parts of the Real Axis that are the Root Locus

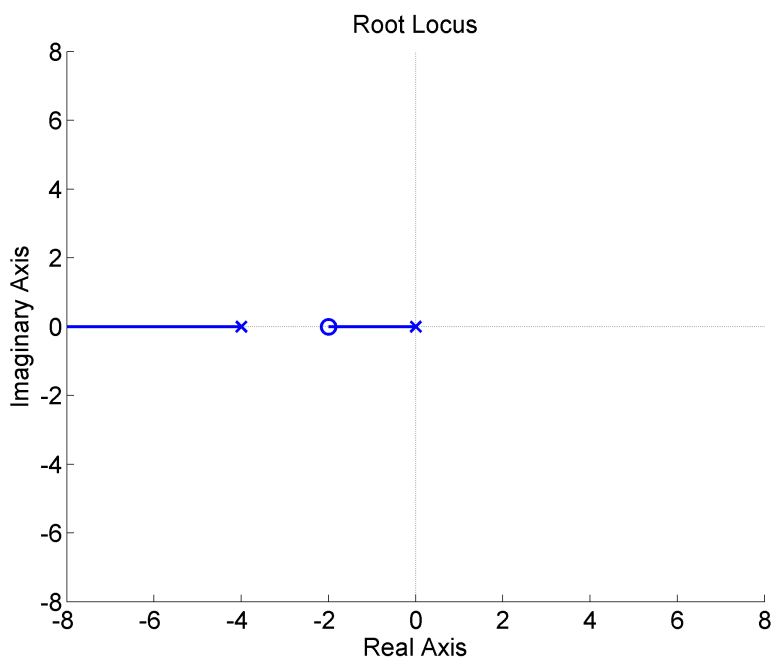
The root locus lies at all points on the real axis to the left of an odd number of poles and zeros that lie on the real axis.

This arises because of the angle criterion (Eq. 5) and the symmetry of the root locus.

For example if we had a characteristic equation of

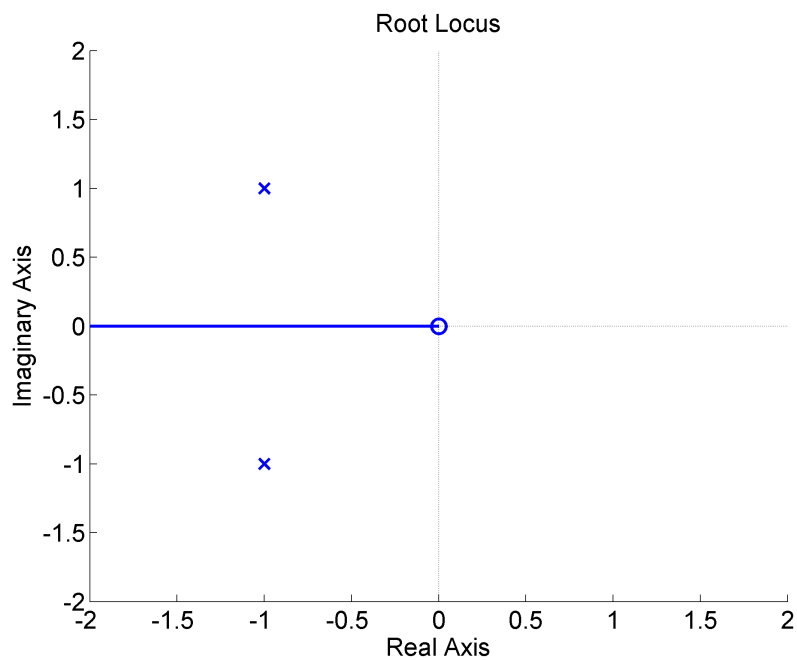
$$1 + K \frac{s + 2}{s(s + 4)} = 0$$

in which $P(s)$ has a zero at $s = -2$ and poles at $s = 0$ and $s = -4$, then the parts of the root locus marked on the real axis are marked by the solid line in the plot below



Case Study

Revisiting our example from Step 1, we see that we have only one zero on the real axis (at $s = 0$) and so all points on the left of $s = 0$ must lie on the root locus. We add that to the root locus plot.



Step 3: Determine the Asymptotes for the loci tending to zeros at infinity

We know that if we have a characteristic equation $P(s)$ that has more poles N than zeros M then $N - M$ of the root locus branches tend to zeros at infinity. These asymptotes intercept the real axis at a point, σ_A , given by

$$\sigma_A = \frac{\sum_{j=1}^N p_j - \sum_{i=1}^M z_i}{N - M}.$$

In other words

$$\sigma_A = \frac{\sum(\text{finite poles}) - \sum(\text{finite zeros})}{N - M}$$

The angles of the asymptotes ϕ_k are given by

$$\phi_A = \frac{2k - 1}{N - M}\pi \text{ for } k = 1, 2, \dots, N - M$$

So for a 4th Order System with a characteristic equation

$$1 + K \frac{s + 1}{s(s + 2)(s + 3)(s + 5)} = 0$$

with 1 zero and 4 poles, 3 branches of the root locus will tend to zeros at infinity. The centre of the 3 asymptotes is

$$\sigma_A = \frac{(0 - 2 - 3 - 5) - (-1)}{4 - 1} = \frac{-9}{3} = 3$$

and have an angles at

$$\phi_A = \frac{2k - 1}{3}\pi \text{ for } k = 1, 2, 3$$

Therefore the angles of the asymptotes are $\pi/3$, π and $5\pi/3 = -\pi/3$. The asymptotes are marked for this system are shown in Fig. 4 in the left plot and are superimposed on the full root locus for this system shown on the right.

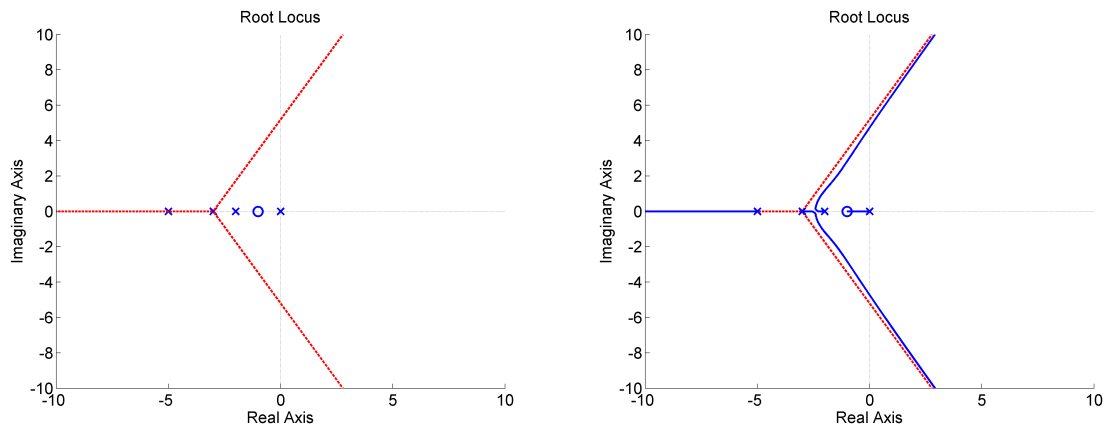


Figure 4: Left: Asymptotes shown for the above example. Right: Asymptotes superimposed on the completed root locus.

Case Study

In the case study example we have $N - M = 2 - 1 = 1$.

Therefore, there is only one asymptote that has an angle

$$\phi_A = \frac{2 - 1}{1} \pi = \pi = 180^\circ$$

and “intercepts” the real axis at

$$\sigma_A = \frac{((-1 + j) + (-1 - j) - (0))}{1} = 2$$

As the asymptote is parallel to the real axis it does not intersect with it. Hence the value of σ_A is meaningless. However, it does lie on the real axis.

A casual glance at the partially completed root locus shows that step 3 merely confirms the result of step 2. All points along the real axis to the left of $s = 0$ are on the root locus.

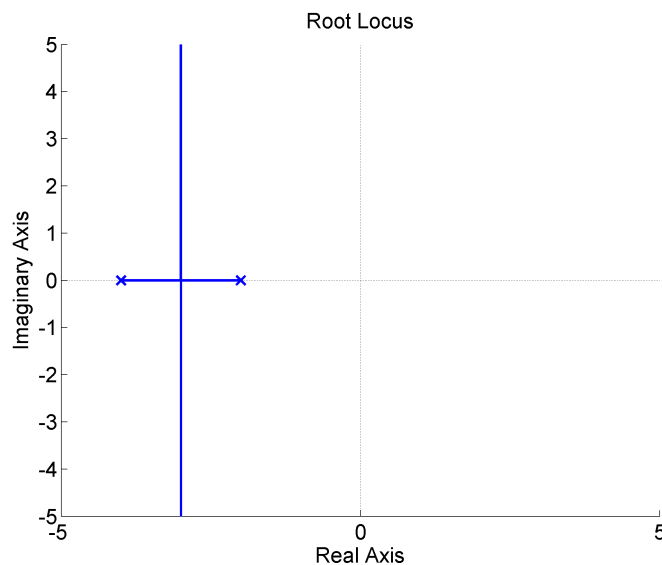
Step 4: Determining the Breakaway Points

From the examples we have seen so far, we can see that often is a point on the real axis at which the root locus either breaks off or onto the real axis.

Considering the root locus of a system with

$$P(s) = \frac{1}{(s + 2)(s + 4)}$$

shown below, we can see that there is a breakaway point on the real axis at a value of $s = \sigma = -3$ and that the angle of the branches as they leave the imaginary axis is perpendicular to the real axis. It turns out that $K = 1$ at the breakaway point.



This corresponds from the system transitioning from being overdamped $K < 1$ to being underdamped $K > 1$ and so marks a dramatic transition in the transient response of the system.

It is possible to determine the breakaway point as follows.

Firstly, identify the portions of the real axis where a breakaway point must exist. Assuming we have already marked the segments of the real axis that are on the root locus, we need to find the segments that are bookended by either two poles or two zeros (either finite zeros or zeros at infinity).

To estimate the values of s at the breakaway points, the characteristic equation

$$1 + KP(s) = 0 \tag{7}$$

is rewritten in terms of K as

$$K = -\frac{1}{P(s)} = K(s).$$

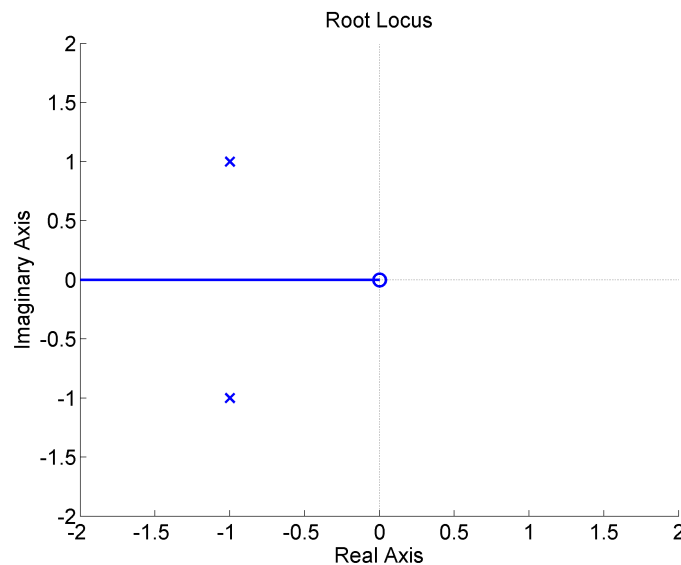
To find the breakaway points we find the values of s corresponding to the maxima in $K(s)$. *i.e.* where

$$\frac{dK}{ds} = K'(s) = 0.$$

As the last step we check the roots of $K'(s)$ that lie on the real axis segments of the locus. The roots that lie in these intervals are the breakaway points.

Case Study

Revisiting our example, after stage 3 we have a partially completed root locus shown below.



There is one segment of the real axis on the root locus. As it is bookended by a zero at $s = 0$ and a zero at infinity ($s = -\infty$) it must contain a breakaway point.

Since

$$P(s) = \frac{s}{s^2 + 2s + 2}$$

we have breakaway points where

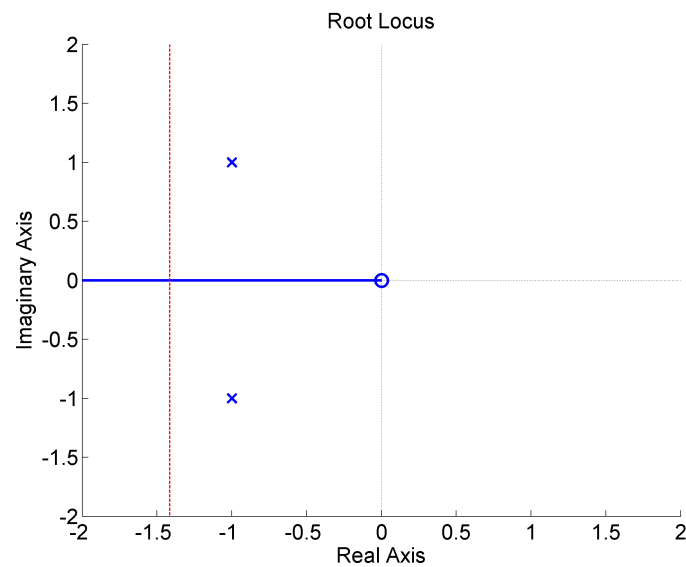
$$\begin{aligned} K'(s) &= \frac{d}{ds} \left(-\frac{s^2 + 2s + 2}{s} \right) = 0. \\ &= -\frac{s(2s + 2) - (s^2 + 2s + 2)}{s^2} \\ &= -\frac{s^2 - 2}{s^2} = 0 \end{aligned}$$

The roots $K'(s)$ are $\pm\sqrt{2}$. As the root at $\sqrt{2}$ does not fall on the root locus segment, the breakaway point occurs at $s = -\sqrt{2}$.

The value of K at the breakaway point is

$$\begin{aligned} K(s) &= -\frac{1}{P(\sqrt{2})}. \\ &= -\frac{\sqrt{2}^2 + 2\sqrt{2} + 2}{\sqrt{2}} \\ &= -\frac{4 + 2\sqrt{2}}{\sqrt{2}} \approx 4.83 \end{aligned}$$

Therefore, we can update the root locus by marking the breakaway point. As the root locus will leave the breakaway points at an angle perpendicular to the real axis. We add this tangent to the root locus sketch.



Step 5: Find the angles of departure/arrival for complex poles/zeros

When there are complex poles or zeros of $P(s)$, the root locus branches will either depart or arrive at an angle θ where, for a complex pole or zero at $s = p$ or $s = z$,

$$\begin{aligned} \angle P^*(p) - \theta &= \pi \text{ for } s = p \\ \text{or } \angle P^*(z) + \theta &= \pi \text{ for } s = z \end{aligned}$$

where

$$\begin{aligned} P^*(p) &= \frac{\prod_{i=1}^M (p - z_i)}{\prod_{\substack{j=1 \\ p \neq p_j}}^N (p - p_j)} \\ P^*(z) &= \frac{\prod_{\substack{i=1 \\ z \neq z_i}}^M (z - z_i)}{\prod_{j=1}^N (z - p_j)} \end{aligned}$$

In other words

$$\begin{aligned} P^*(p) &= (s - p)P(s) \Big|_{s=p} \\ P^*(z) &= \frac{P(s)}{s - z} \Big|_{s=z} \end{aligned}$$

Therefore, exploiting the rules of complex numbers, we can rewrite $\angle P^*(p)$ and $\angle P^*(z)$ as

$$\begin{aligned} \angle P^*(p) &= \sum_{i=1}^M \angle(p - z_i) - \sum_{\substack{j=1 \\ p \neq p_j}}^N \angle(p - p_j) \\ \angle P^*(z) &= \sum_{\substack{i=1 \\ z \neq z_i}}^M \angle(z - z_i) - \sum_{j=1}^N \angle(z - p_j) \end{aligned}$$

Case Study

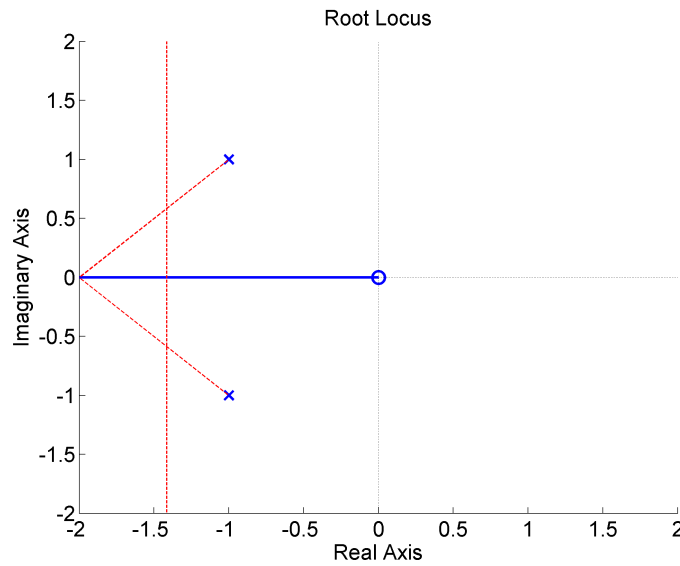
Here we have two complex poles at $s = -1 \pm j$.

Therefore, for the pole at $s = -1 + j$, the root locus departs the pole at an angle θ where

$$\begin{aligned}
 \angle P^*(-1 + j) - \theta &= \pi \\
 \Rightarrow \angle (s + 1 - j)P(s) \Big|_{s=-1+j} - \theta &= \pi \\
 \Rightarrow \angle \frac{s}{s - (-1 - j)} \Big|_{s=-1+j} - \theta &= \pi \\
 \Rightarrow \angle -1 + j - \angle -1 + j - (-1 - j) - \theta &= \pi \\
 \Rightarrow \frac{3\pi}{4} - 0 - \frac{\pi}{2} - \theta &= \pi \\
 \Rightarrow \theta &= -\frac{3\pi}{4}
 \end{aligned}$$

We could repeat this calculation to estimate θ for $s = -1 - j$. However, exploiting the symmetry of the root locus, it is quicker to notice that the value of $\theta_{-1-j} = -\theta_{-1+j} = \frac{3\pi}{4}$.

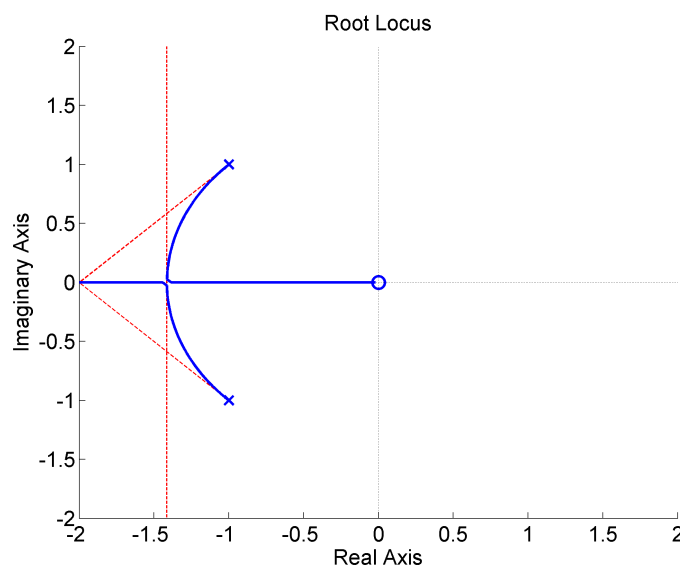
Therefore we can update the root locus plot by adding tangents corresponding to the angles of departure for the complex poles.



Step 6: Complete the Sketch

This can be done by hand from all the information gained from the previous steps. You could make it simpler by finding the roots of the characteristic equation for more values of K and placing them on the locus.

The root locus for our case study is shown below.



3.1 Summary

The process of generating the root locus for $0 \leq K < \infty$ can be summarised as follows

Step	Rule
Estimate the transfer function and write the characteristic equation of that the tuning parameter, K , appears as a multiplier.	$1 + KP(s) = 0$
Mark the start and end points of the loci.	Find the poles (start points) and zeros (end points) of $P(s)$.
Determine the number of separate loci	It is the number of poles of $P(s)$.
Locate the segments of the real axis that are root loci	Locus lies to the left of an odd number of zeros or poles.
Find the asymptotes of the loci tending to zeros at infinity	The intersection point of the asymptotes is $\sigma_A = \frac{\sum p_j - \sum z_i}{N-M}$. The angles are $\phi_A = \frac{2k-1}{N-M}\pi$
Determine the breakaway points.	Set $K(s) = -\frac{1}{P(s)}$ and find $K'(s) = 0$
Determine the angles of arrival/departure at complex zeros/poles of $P(s)$	For poles the angle of departure is $\angle P^*(p_k) - \pi$. For zeros the angle of arrival is $\pi - \angle P^*(z_k)$
Complete the Sketch	

4 Root Locus for Negative K

A similar set of rules exists for plotting the root locus for $0 \geq K > -\infty$. The summary for plotting the negative K root is shown below with differences from the positive K root locus highlighted in bold.

Step	Rule
Estimate the transfer function and write the characteristic equation of that the tuning parameter, K , appears as a multiplier.	$1 + KP(s) = 0$
Mark the start and end points of the loci.	Find the poles (start points) and zeros (end points) of $P(s)$.
Determine the number of separate loci	It is the number of poles of $P(s)$.
Locate the segments of the real axis that are root loci	Locus lies to the left of an even number of zeros or poles.
Find the asymptotes of the loci tending to zeros at infinity	The intersection point of the asymptotes is $\sigma_A = \frac{\sum p_j - \sum z_i}{N-M}$. The angles are $\phi_A = 2\frac{2k-1}{N-M}\pi$
Determine the breakaway points	Set $K(s) = -\frac{1}{P(s)}$ and find $K'(s) = 0$
Determine the angles of arrival/departure at complex zeros/poles of $P(s)$	For poles the angle of departure is $\angle P^*(p_k)$. For zeros the angle of arrival is $\angle P^*(z_k)$
Complete the Sketch	

Case Study

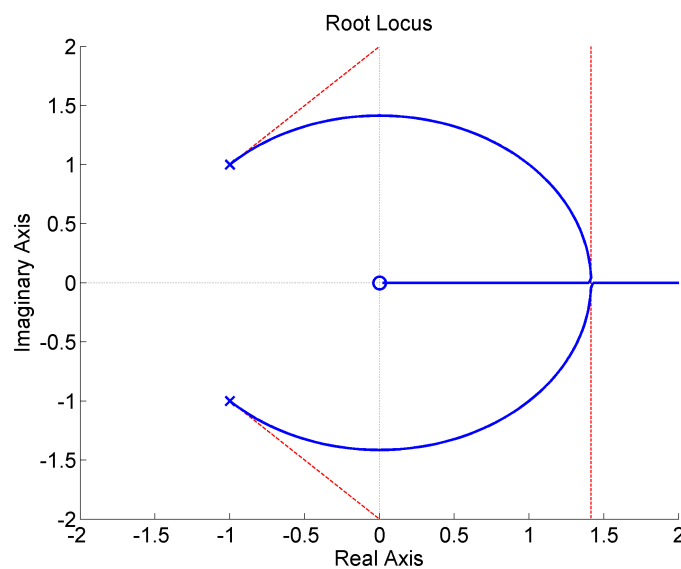
We will now plot the negative K root locus for a system with

$$H(s) = \frac{Ks}{s^2 + (2 + K)s + 2}.$$

Hence,

$$P(s) = \frac{s}{s^2 + 2s + 2}.$$

The root locus is plotted below.



Notice that for a value of $K < -2$ the roots move into the RHS of the s-plane and so the system is unstable.

5 Conclusion

In this handout we showed how to determine the location of the poles of a control system through a root locus plot. These plots are important because they indicate whether a system is stable or not and can also be used to estimate the transient response.

6 Design Case Study: A Laser Control System

A Laser system is used to shape the hip socket for the appropriate insertion of an artificial hip joint. The system uses a motorised system for controlling the position of the laser using a closed loop control system with a proportional controller. The block diagram for the system is

As this system is to be used in surgery it requires high accuracy for the position and velocity responses. In order to get the required velocity response the controller gain, K , must be adjusted so that the steady state error for a ramp input signal $x(t) = At$ (where $A = 1$ mm/s) is less than 0.1 mm. The system must also be stable. A root locus plot of this system for $0 \leq K < \infty$ is shown in Fig ??.

1. Determine the value of K that satisfies the steady state error and the stability constraints.
2. Determine the position of the poles when the steady state error constraint is met exactly.
3. Assuming that the system contains a pair of dominant complex poles, find the settling time and percentage overshoot to a unit step input.
4. If the actual system response to a unit step is shown in Fig ??, measure the percentage overshoot and settling time to determine whether it is safe to assume that the system has dominant poles.