

Characterising the Response of a Closed Loop System

Signals and Systems: 3C1
Control Systems Handout 2
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- In the last handout we showed that using a closed loop system allows us to control the the behaviour output signal. It allows us to tune both the transient and steady state responses of the output.
- We used the the example of the cruise controller to illustrate this. To get a rise time of less than 5 seconds and a steady state error of 2% we needed a gain parameter of greater than 2450.
- In this handout, we will look at in more detail the ways in which the response can be controlled. Specifically, we will look at a parameterisation of the response that will allow us to dictate a set of design specifications for the behaviour of the system. This parameterisation depends on the order of the system (*i.e.* The number of poles) as well as the form of the test input signal.
- **We will spend most time looking at the response of a second order system to an input step signal.** We will define expressions for

the response parameters and will show how the design specifications affect the possible location of poles on the s-plane.

- We will also look at **PID Controllers** which are the most commonly used class of controllers and we will investigate how they affect the transient and steady-state responses to a step input.
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1 Test Input Signals

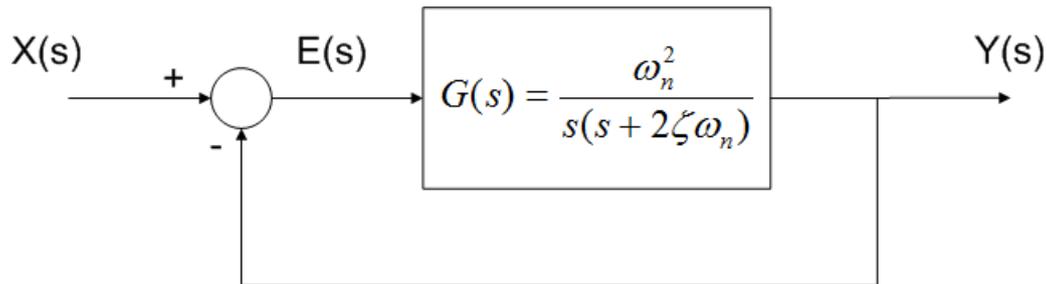
In practice, the actual input signal to a control system will be unknown. For example, an automated vehicle control system can be designed so that the speed of the vehicle matches the speed of the vehicle in front. In this system the input signal is the speed of the vehicle in front, which is dictated by the behaviour of the driver.

However, when designing control systems we use a set of standard test signals to model the input. This approach is useful as these signals can approximate well the actual behaviour. Furthermore they allow us to compare the performance of competing controller designs. The most commonly used test signals are the step, ramp and parabolic functions.

Test Signal	$x(t)$ for $t > 0$	Usage Scenario (for vehicle control systems)
Step	A	A system where the speed follows a constant reference speed.
Ramp	At	The speed matches the vehicle in front which is undergoing a constant acceleration.
Parabolic	At^2	A system where the car maintains a constant distance from the car in front which is under a constant acceleration.

2 Performance of 2nd Order Systems

We will now examine a closed loop 2nd Order System and determine its response to a unit step function $x(t)$. Consider the following system



The transfer function for this system is calculated as

$$\begin{aligned}
 H(s) &= \frac{Y(s)}{X(s)} \\
 &= \frac{(X(s) - Y(s)) G(s)}{X(s)} \\
 &= G(s) - \frac{Y(s)}{X(s)} G(s) = G(s) - H(s)G(s) \\
 \Rightarrow H(s) &= \frac{G(s)}{1 + G(s)}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 H(s) &= \frac{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}}{1 + \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}} \\
 &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
 \end{aligned}$$

For a unit step input we have $X(s) = 1/s$ and so the output Laplace transform is

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Using a partial fraction expansion we can show that for $0 < \zeta < 1$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta)$$

where $\beta = \sqrt{1 - \zeta^2}$ and $\theta = \cos^{-1}(\zeta)$.

A graph of the step response is shown in Fig. ???. When $\zeta < 1$ we see that the output has an oscillatory behaviour, since the poles of the transfer function are complex. When $\zeta > 1$, the system has real poles and hence the system is overdamped and the response has no oscillatory behaviour.

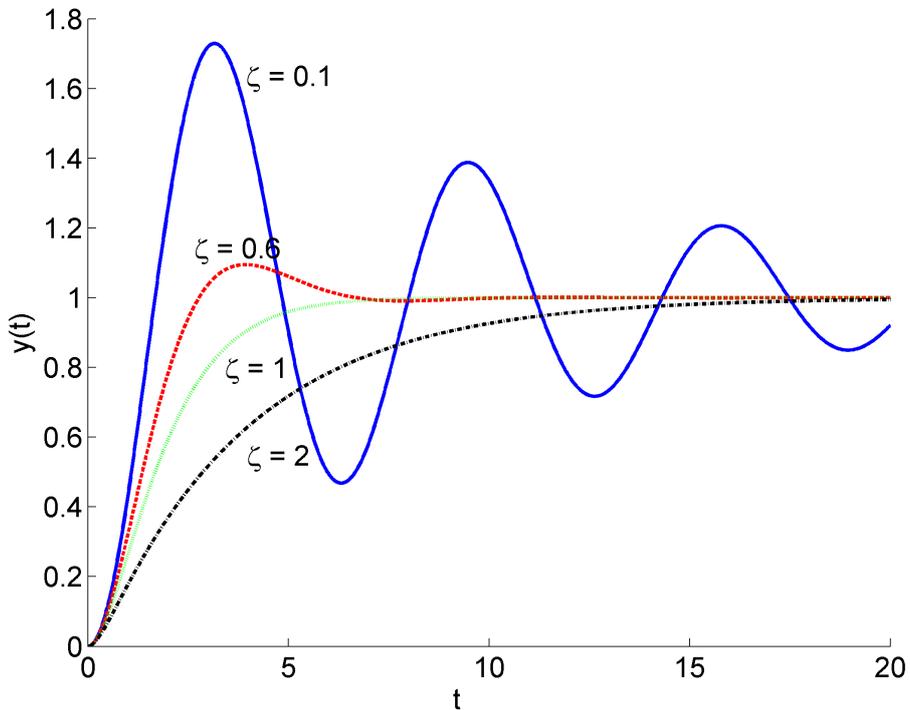


Figure 1: A plot of the step response of 4 different values of ζ .

2.1 Parameterisation of the 2nd Order Step Response

In order to obtain the desired step response that that is required, we chose a set of meaningful parameters that define the behaviour of the response. The most commonlt chosen parameters are shown in Fig. ??.

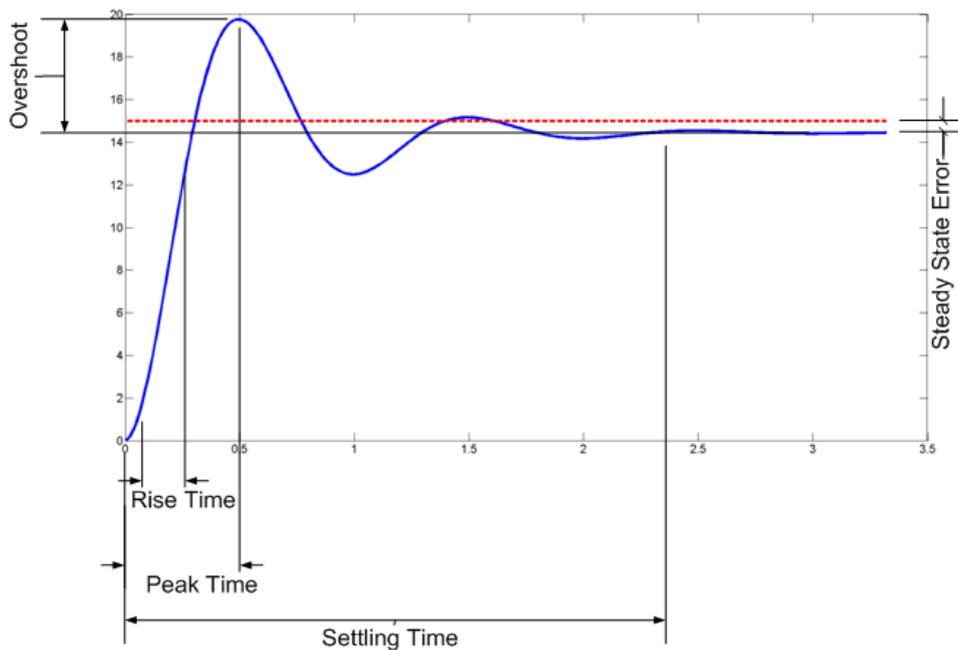


Figure 2: A typical step response for a 2nd order system with complex poles and no finite zeros.

- Rise Time (t_r) - the time taken for the output to go from 10% to 90% of the final value.
- Peak Time (t_p) - the time taken for the output to reach its maximum value.
- Overshoot - $(\text{max value} - \text{final value})/\text{final value} \times 100$.
- Settling Time (t_s)- The time taken for the signal to be bounded to within a tolerance of $x\%$ of the steady state value.
- Steady State Error e_{ss} - The difference between the input step value (dashed line) and the final value.

Note that if the system contains no complex poles then the response will not be oscillatory and hence the peak time and overshoot will not be relevant.

Typically when designing a control system, we will be asked to achieve certain targets for these parameters. For example design a suitable controller such that the rise time is less than 3 seconds and a steady state error of 0. To do this, it is necessary to either run a simulation and measure the parameters from the step response directly or to define expressions for the parameters in terms of the transfer function coefficients.

For a 2nd System it is possible to write down expressions for the parameters of the step response (for $0 < \zeta < 1$) as follows.

$$\text{Overshoot \%} = 100 \times \exp \left\{ \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \right\} \text{ for } 0 < \zeta < 1$$

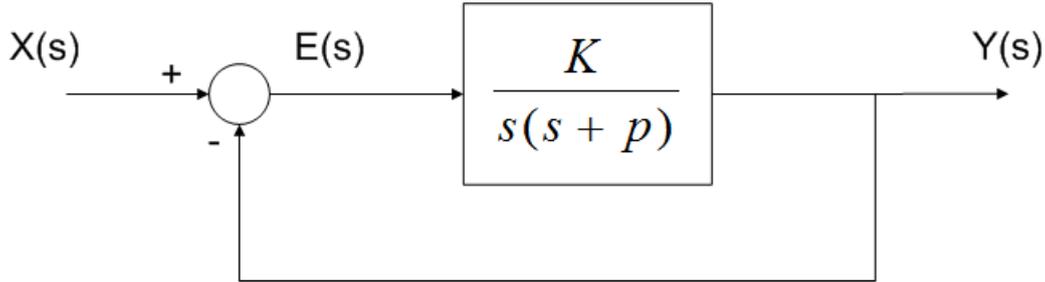
$$t_r \approx \frac{1}{\omega_n} (2.3\zeta^2 - 0.078\zeta + 1.12) \text{ for } 0 < \zeta < 1$$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$t_s \approx -\frac{\ln(\text{tolerance})}{\zeta\omega_n} \text{ for } \zeta \ll 1$$

Example: Parameter Selection

For the closed loop control system shown choose the gain value K and parameter p so that for a step input the percentage overshoot is less than 5% and the response settles to within 2% of its final value within 4 seconds.



The transfer function for this system is

$$H(s) = \frac{K}{s^2 + ps + K}$$

Therefore we have

$$\begin{aligned} p &= 2\zeta\omega_n \\ K &= \omega_n^2 \end{aligned} \quad (1)$$

For an overshoot of less than 5% we have

$$\begin{aligned} \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) &< 0.05 \\ \Rightarrow -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} &< \ln(0.05) = -3.00 \\ \Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} &> 0.95 \\ \Rightarrow \frac{\zeta^2}{1-\zeta^2} &> 0.91 \\ \Rightarrow \zeta^2 &> \frac{0.91}{1+0.91} = 0.48 \\ \Rightarrow \zeta &> 0.69 \end{aligned}$$

and for a settling time (2% error) of less than 4 seconds

$$t_s = \frac{\ln(0.02)}{\zeta\omega_n} < 4$$
$$\Rightarrow \zeta\omega_n > 1$$

As $p = 2\zeta\omega_n$ we need to choose $p > 2$. Also we need to choose $K = \omega_n^2 > 1/\zeta^2$.

So for example if we choose $\zeta = 0.7$ to meet the overshoot criteria, then

$$K > \frac{1}{0.7^2} \approx 2.$$

It is often instructive if we think about what restrictions these design criteria place on the locations of the poles of the second order system. Recall that for a second order system with $\zeta < 1$ the poles p_i and p_i^* are given by

$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

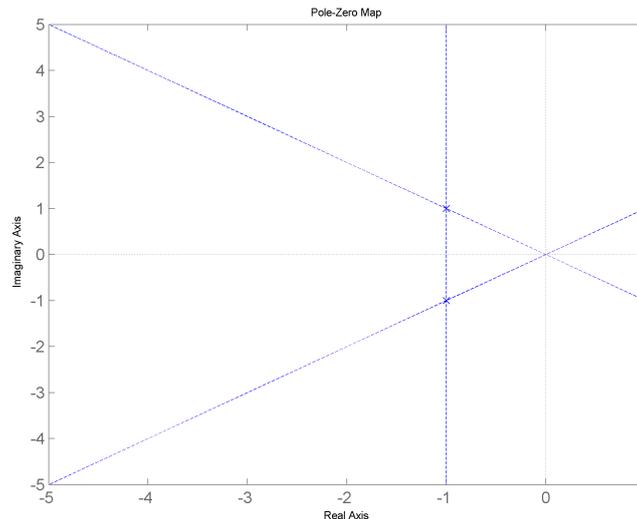
Therefore the real part of the pole is given by $-\zeta\omega_n$ which by implication must be less than -1 to achieve the required settling time. Recall also that ζ is the cosine of the angle θ where

$$\theta = \pi - \arg(p_i).$$

Hence to meet the overshoot specification

$$\theta < \cos^{-1}(0.69) \approx \frac{\pi}{4}.$$

Therefore, these criteria restrict the area of the s-plane on which the poles can lie.



2.2 Effect of Adding an Extra Pole or Zero on the Transient Response

The values for these parameters (t_r , t_s etc.) are only valid for 2^{nd} Order Systems without finite zeros. If extra poles or zeros are added to the system then the step response would be different and we would have to derive new expressions for the parameters. However, many systems possess a pair of dominant roots, in which case the step response will be approximated by a 2^{nd} Order Systems without finite zeros.

Example: A 3^{rd} Order System

Consider the case where an extra real pole is been added to a 2^{nd} Order Systems without finite zeros.

$$H(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(\gamma s + 1)} \quad (2)$$

Thus the extra pole has a value of $-1/\gamma$. If this pole is far away enough from the imaginary axis (*i.e.* $1/\gamma$ is large enough) then the effect of the extra pole on the step response can be ignored. This can be seen in Fig. ?? where the step response is plotted for different values of γ . As γ gets smaller the shape of the step response tends to the step response of the 2^{nd} Order System. This is confirmed by comparing the overshoot and settling times for each plot (Table ??).

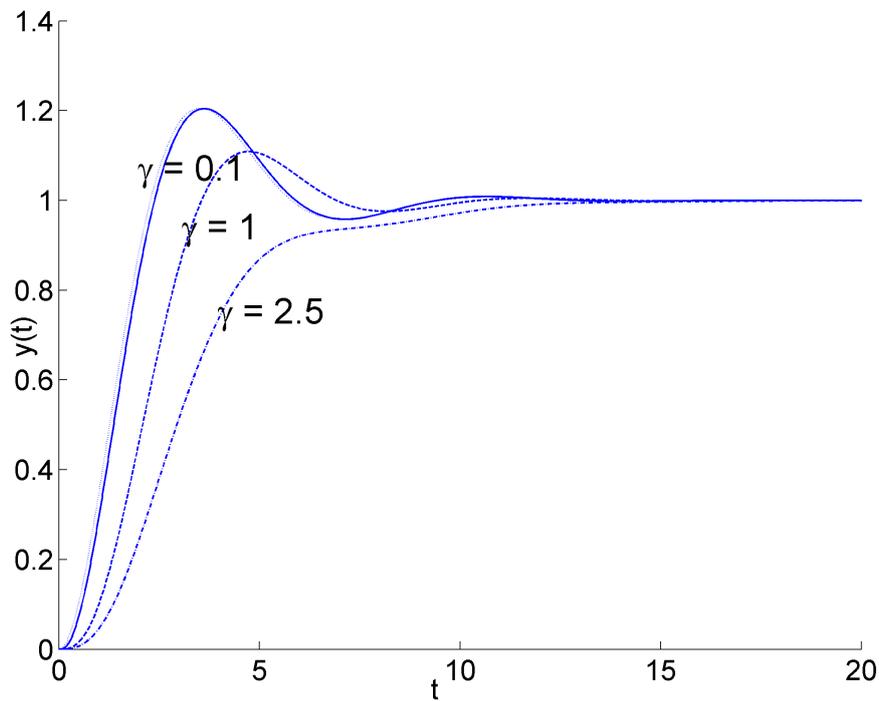


Figure 3: A plot of the step response versus γ . The finely dotted line represents the response of the 2nd Order System. In this example $\omega_n = 1$ and $\zeta = 0.45$.

Example: A 2nd Order System with a finite zeros

Consider a 2nd order system with a finite zero. It has a transfer function of the form

$$H(s) = \frac{\omega_n^2 (\tau s + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3)$$

Similarly to the case of an extra pole, the effects of the extra zero are reduced the further away it lies from the imaginary axis. This is shown in Fig. ?? and Table ??.

pole value = $1/\gamma$	Percentage Overshoot	Settling Time
-0.44	0	9.63
-0.66	3.9	6.3
-1.11	12.3	8.81
-2.5	18.6	8.67
-20.0	20.0	8.37
$-\infty$	20.5	8.24

Table 1: Comparing the Percentage Overshoot and 2% settling time for different 3rd pole locations. The last row corresponds to the 3rd pole at infinity. This means the system is 2nd Order.

zero value = $1/\tau$	Percentage shoot	Over- Peak Time	Settling Time
-0.4	10	1.7	10.1
-1	36.0	2.4	7.58
-10	20.6	3.4	8.24
$-\infty$	20.5	3.5	8.24

Table 2: Comparing the Percentage Overshoot and 2% settling time for different finite zero locations. The last row corresponds to a zero at infinity (*i.e.* no finite zeros).

Summary

- This analysis shows that is sometimes possible to ignore the effects of poles and zeros when considering the system response. Such systems are said to contain a set of dominant poles which overwhelmingly determine the transient parameters of the system response. This can ease analysis of complicated systems.
- Even if there are no dominant poles it is also possible to approximate higher order systems as 2nd Order. WE DO NOT COVER THIS IN 3C1.
- In practice it is possible to simulate the response of higher order systems using software simulation (*e.g.* in MATLAB). Therefore, making these

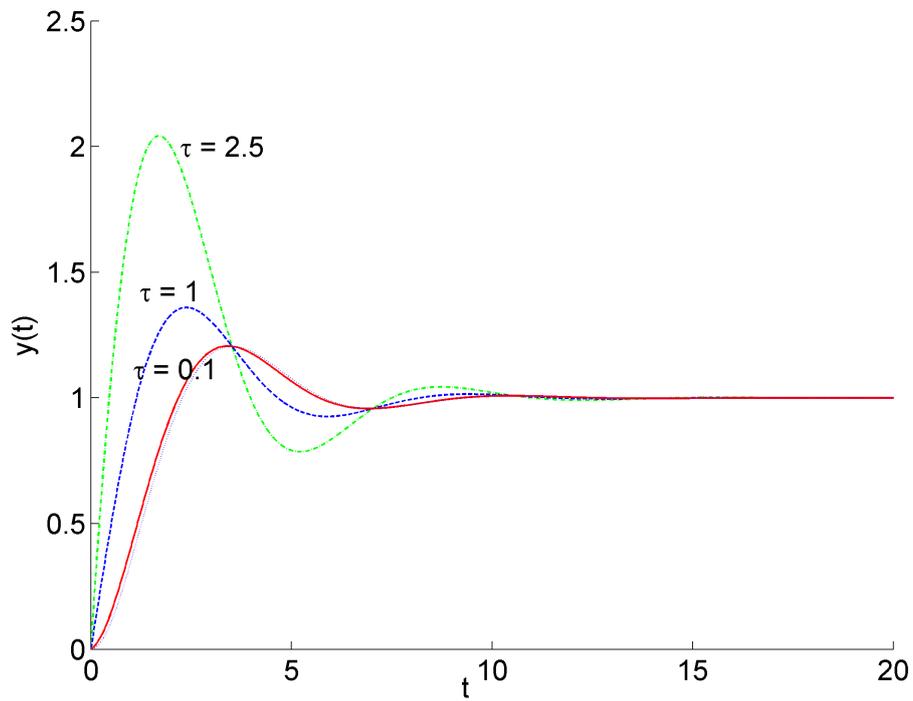
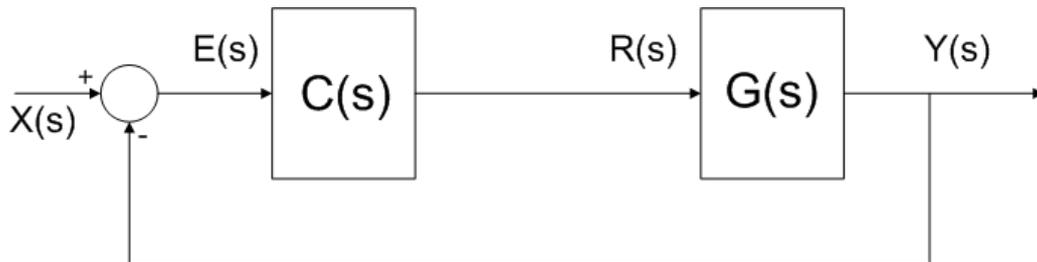


Figure 4: A plot of the step response versus τ . The finely dotted line represents the response of the system without a finite zero. In this example $\omega_n = 1$ and $\zeta = 0.45$.

type of simplifications to the transfer functions is often unnecessary.

2.3 Steady State Error Performance

As well being able to adjust the transient response, we saw in the last handout that closed loop systems are less sensitive to steady state errors than open loop systems. To estimate the steady state error, we estimate the steady state of the signal corresponding to $E(s)$ in the block diagram below



For this control system architecture the value of $E(s)$ is

$$E(s) = \frac{1}{1 + C(s)G(s)}X(s).$$

Therefore, according to the final value theorem the steady state error is

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{1}{1 + C(s)G(s)}X(s).$$

For a given plant $G(s)$, the value of e_{ss} is dictated by the controller $C(s)$ as well as the choice of input signal. Assuming $G(s)$ is a rational transfer function we can write the general form of the transfer function as

$$C(s)G(s) = \frac{K n'(s)}{s^N d'(s)}, \quad (4)$$

where all the roots of the polynomials $n(s)$ and $d(s)$ are non-zero. Intuitively this is cascade of a system with a series of N integrators. The number of integrators is largely dictated by the choice of the controller and is referred to as the type number of the controller. (*e.g.* If we employ a type-2 Controller then $N = 2$). For a given input signal the system type will dictate whether the steady state error is either zero, finite or infinite.

Aside A system with a transfer function $H(s) = 1/s$ is referred to as an integrator. This is because we can rewrite the characteristic equation of the system as $sY(s) = X(s)$ and so taking inverse Laplace Transforms we get

$$\begin{aligned}\frac{dy}{dt} &= x(t) \\ \Rightarrow y(t) &= \int x(t)dt.\end{aligned}$$

When a system has a transfer function $H(s) = s$ then the system is referred to as a differentiator. This is because for such a system the output Laplace transform is

$$\begin{aligned}Y(s) &= sX(s) \\ \Rightarrow y(t) &= \frac{dx(t)}{dt}.\end{aligned}$$

Note that because a transfer function $H(s) = s$ it is not physically realisable as it has more zeros than poles. In practice differentiators have a transfer function

$$H(s) = \frac{s}{\gamma s + 1}.$$

However, if we design the differentiator so that the pole at $-1/\gamma$ is far enough away from the imaginary axis then we can say that $H(s) \approx s$.

Step Input

The steady state error for a step input $Au(t)$ is

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s(A/s)}{1 + C(s)G(s)} = \frac{A}{1 + \lim_{s \rightarrow 0} C(s)G(s)}.$$

Therefore, the value of e_{ss} is dictated by the value $\lim_{s \rightarrow 0} C(s)G(s)$. Referring back to equation ??, if we have a type-0 system, then

$$\lim_{s \rightarrow 0} C(s)G(s) = K \frac{n'(0)}{d'(0)} = B,$$

where B is finite¹. Therefore,

$$e_{ss} = \frac{A}{1 + B}. \quad (5)$$

Consequently, the steady state error is finite (and non-zero) for a type-0 system.

For $N \geq 1$, then

$$\lim_{s \rightarrow 0} C(s)G(s) = \frac{K n'(0)}{0^N d'(0)} = \infty,$$

therefore

$$e_{ss} = \frac{A}{1 + \infty} = 0. \quad (6)$$

Therefore, the steady state error to a step input is 0 for a system of type-1 or higher.

¹B is finite because the roots of $n(s)$ and $d(s)$ are non-zero.

Ramp Input

The steady state error for a step input At is

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s(A/s^2)}{1 + C(s)G(s)} \\ &= \lim_{s \rightarrow 0} \frac{A}{s(1 + C(s)G(s))} \\ &= \lim_{s \rightarrow 0} \frac{A}{s + sC(s)G(s)} \end{aligned}$$

For type-0 systems, the steady state error is infinite as

$$e_{ss} = \frac{A}{0 + \lim_{s \rightarrow 0} sC(s)G(s)} \quad (7)$$

and from equation ??

$$\begin{aligned} \lim_{s \rightarrow 0} sC(s)G(s) &= \lim_{s \rightarrow 0} s \frac{n'(s)}{d'(s)} \\ &= 0 \times B = 0 \\ \Rightarrow e_{ss} &= \frac{A}{0 + 0} = \infty. \end{aligned}$$

For a type-1 system,

$$\begin{aligned} e_{ss} &= \frac{A}{0 + \lim_{s \rightarrow 0} s \frac{K n'(s)}{s d'(s)}} \\ &= \frac{A}{0 + K \frac{n'(0)}{d'(0)}} = \frac{A}{B}. \end{aligned}$$

Hence, the steady state error is finite. For a type-2 system or higher we can repeat the analysis to show that $e_{ss} = 0$

Parabolic Input

For a parabolic input of the form $x(t) = At^2/2$ (*i.e.* $X(s) = A/s^3$)

$$\begin{aligned} e_{ss} &= \lim_{s \rightarrow 0} \frac{s(A/s^3)}{1 + C(s)G(s)} \\ &= \lim_{s \rightarrow 0} \frac{A}{s^2 + s^2 C(s)G(s)} \\ &= \frac{A}{\lim_{s \rightarrow 0} s^2 C(s)G(s)}. \end{aligned}$$

So for a type 1 or type 0 system the expression of e_{ss} is of the form

$$e_{ss} = \frac{A}{K \lim_{s \rightarrow 0} s^k \frac{n'(s)}{d'(s)}}$$

where $k > 0$ and so the steady state error is infinite. For type-2 systems $k = 0$ and e_{ss} is finite ($= A/B$). For type-3 systems of higher $k < 0$ and hence there is zero steady state error.

Table ?? summaries presents a summary of these results.

Type Number	Step Input $x(t) = Au(t)$	Ramp Input $x(t) = At$	Parabolic Input $x(t) = At^2/2$
0	$\frac{A}{1+B}$	∞	∞
1	0	$\frac{A}{B}$	∞
2	0	0	$\frac{A}{B}$

Table 3: Comparing the steady state errors for different system types and input signals.

3 PID Controllers

PID Controllers are the most widely used class of controllers used in control systems. The transfer function of a PID controller is

$$C(s) = K_p + \frac{K_i}{s} + K_d s. \quad (8)$$

Given the input error signal $e(t)$ the output signal $r(t)$ (*i.e.* $R(s) = C(s)E(s)$) is

$$r(t) = K_p e(t) + K_i \int e(t) dt + K_d \frac{de(t)}{dt}.$$

This equation explains the name PID which is short for Proportional-plus-Integral-plus-Derivative control. The gain factors K_p , K_i and K_d represent the relative weighting towards the proportional, integral and derivative terms respectively.

If we set $K_i = 0$, then the controller is a proportional-plus-integral (PI) controller and has a transfer function

$$C(s) = K_p + \frac{K_i}{s}.$$

If $K_d = 0$, then the controller is a proportional-plus-derivative (PD) controller.

$$C(s) = K_p + K_d s.$$

If both $K_i = 0$ and $K_d = 0$, the controller is a proportional controller.

$$C(s) = K_p.$$

PID controllers are popular as they are suitable for a wide variety of applications and require the adjustment of only 3 parameters, making relatively simple to tune. The effect of increasing each of the parameters is summarised in the table below.

Control Action	Rise Time	Overshoot	Settling Time	Steady State Error
Increasing K_p	reduces	increases	small change	reduces
Increasing K_i	reduces	increases	increases	eliminates
Increasing K_d	small change	reduces	reduces	small change

Table 4: The effect of increasing each type of control action on the impulse response.

Increasing the proportional gain (K_p) will have the effect of reducing the rise time and will reduce, but never eliminate, the the steady-state error. The steady state error can be eliminated by introducing some element of integral control (K_i). However, it may make the transient response worse. Adding derivative control (K_d) will have the effect of improving the transient response without impacting the steady state response. However, adding derivative control makes the system more sensitive to measurement noise (in the transducer) which can affect the stability of the controller.

When, designing a control system the three gain parameters must be tuned individually to achieve the desired response. There are a number of standardised methodologies for tuning which can be used to simplify the process but often tuning is relies on trial-and-error as well as the experience of the engineer.

Note Table ?? represents behaviour when altering the PID gain values. It is possible to find examples where some of the effects listed in the table do not occur. The exact effect will depend on the order of the system, the locations of the system poles and zeros as well the value of the other PID parameters.

3.1 Case Study: Car Cruise Control using a PI Controller

Recall that we have a closed loop car cruise controller which we wish to design to meet the following criteria

- A rise time of less than 5 seconds.
- A steady state error of less than 2%.

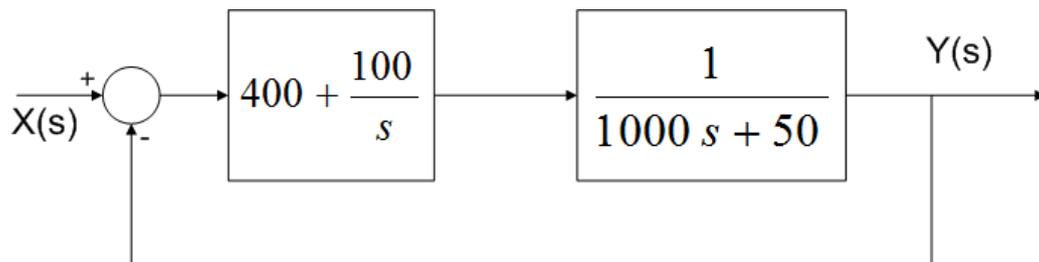
We saw in the last handout that we required a proportional controller gain $K_p > 390$ to satisfy the rise time constraint but that we needed $K_p > 2450$ to satisfy the steady state error condition. We will now show that it is possible to meet both criteria using a PI controller which will eliminate the steady state error completely, allowing us to use smaller values of K_p .

Consider the PI controller

$$C(s) = K_p + \frac{K_i}{s}$$

where $K_p = 400$ and $K_i = 100$.

The block diagram for the cruise controller is



The transfer function is therefore

$$\begin{aligned} H(s) &= \frac{C(s)G(s)}{1 + C(s)G(s)} \\ &= \frac{(400 + 100/s) \frac{1}{1000s+50}}{1 + (400 + 100/s) \frac{1}{1000s+50}} \\ &= \frac{\frac{400s+100}{s(1000s+50)}}{1 + \frac{400s+100}{s(1000s+50)}} \\ &= \frac{400s + 100}{1000s^2 + 450s + 100} \\ &= \frac{0.1(4s + 1)}{s^2 + 0.45s + 0.1} \end{aligned} \tag{9}$$

So we have a 2nd Order System with $\omega_n^2 = 0.1$ and a damping ratio $\zeta \approx 0.7$. Therefore, it is an underdamped system and hence there are two complex poles. The system also has a finite zero at $s = -0.0025$. The pole/zero plot for this system are shown in the pole-zero plot in Fig. ??.

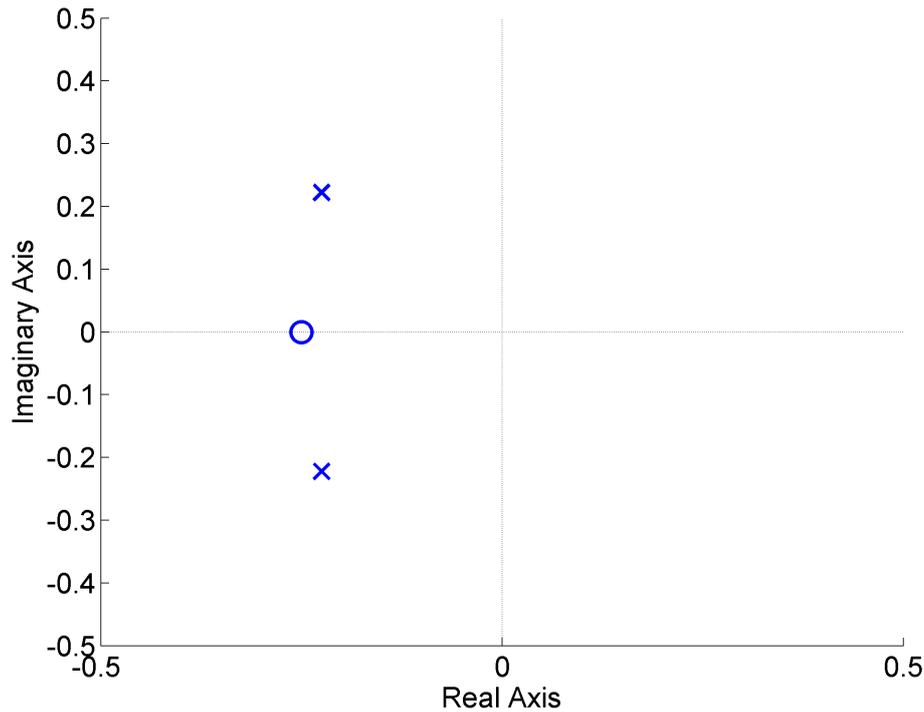


Figure 5: A pole zero plot for the cruise controller with $K_p = 400$ and $K_i = 100$.

Looking at the step response in Fig. ?? we can see that the rise time is slightly greater than 3 seconds and it takes about 30 seconds to reach the desired 10 m/s.

However, overshoot has been introduced into the system. Is this a good idea for a cruise control system? The overshoot can be eliminated by choosing smaller values of $K_i = 1$ but in this case it will take a lot longer to reach the steady state²

²Notice that this contradicts the expected effect on the settling time in Table ??.

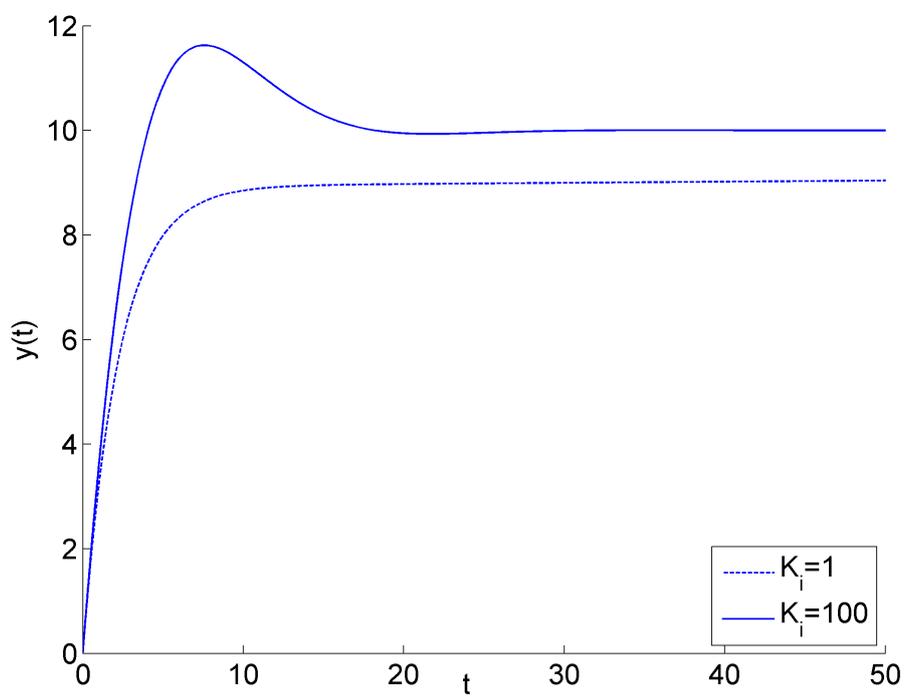


Figure 6: A plot of the step response for the PI controller for $K_i = 100$ and $K_i = 1$. The plot for $K_i = 1$ eventually converges to 10 m/s after about 1000 seconds.