Paper 3C1

Answers for Examples Sheet 8: Digital Systems

David Corrigan 2011

http://www.mee.tcd.ie/~sigmedia

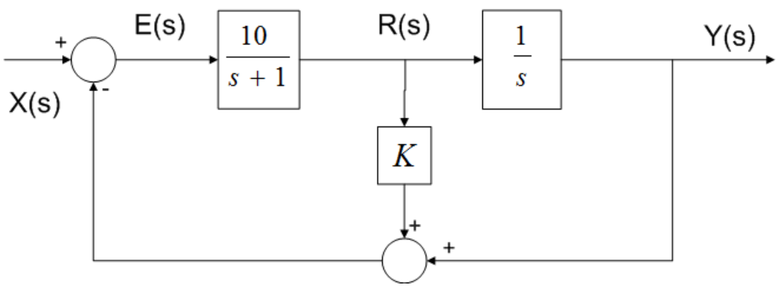
1. (a) Simple Control Systems like this have a transfer function of the form

$$H(s) = \frac{G(s)}{1+G(s)}$$

Therefore

$$\begin{aligned} H(s) &= \frac{\frac{10}{s^2+2s+10}}{1+\frac{10}{s^2+2s+10}} \\ &= \frac{10}{s^2+2s+10} \end{aligned} \quad (1)$$

(b) To solve problems like this it is best to use labels for the ouput of each stage Then we write



down the relationship between each output.

$$\begin{aligned} Y(s) &= \frac{1}{s}R(s) \\ R(s) &= \frac{10}{s+1}E(s) \\ E(s) &= X(s) - KR(s) - Y(s) \end{aligned}$$

Working back from the output towards the input we get

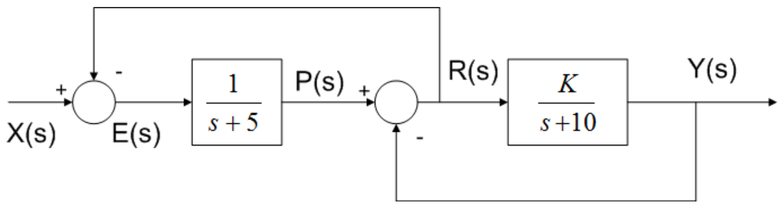
$$Y(s) = \frac{10}{s(s+1)}(X(s) - KR(s) - Y(s))$$

Expressing $R(s)$ in terms of $Y(s)$ we get

$$Y(s) = \frac{10}{s(s+1)}(X(s) - KsY(s) - Y(s))$$

$$\Rightarrow Y(s)\left(1+\frac{10Ks}{s(s+1)}+\frac{10}{s(s+1)}\right)=\frac{10}{s(s+1)}X(s)$$
$$\Rightarrow H(s)=\frac{Y(s)}{X(s)}=\frac{10}{s^2+(10K+1)s+10}$$

(c) Again we label the block diagram Then



$$\begin{aligned}
Y(s) &= \frac{K}{s+10}R(s) \\
R(s) &= P(s) - Y(s) \\
P(s) &= \frac{1}{s+5}E(s) \\
E(s) &= X(s) - R(s)
\end{aligned}$$

Working back from the output we get

$$\begin{aligned}
Y(s) &= \frac{K}{s+10}R(s) \\
&= \frac{K}{s+10}\left(\frac{1}{s+5}E(s) - Y(s)\right) \\
&= \frac{K}{s+10}\left(\frac{1}{s+5}\left(X(s) - R(s)\right) - Y(s)\right) \\
&= \frac{K}{s+10}\left(\frac{1}{s+5}\left(X(s) - \frac{s+10}{K}Y(s)\right) - Y(s)\right) \\
&= \frac{K}{(s+5)(s+10)}X(s) - \frac{1}{s+5}Y(s) - \frac{K}{s+10}Y(s)
\end{aligned}$$

Bringing all the $Y(s)$ terms to the left hand side we get

$$Y(s)\left(1+\frac{1}{s+5}+\frac{K}{s+10}\right)=\frac{K}{(s+5)(s+10)}X(s)$$

Dividing both sides by the factor of $Y(s)$ on the LHS we get

$$\Rightarrow H(s)=\frac{Y(s)}{X(s)}=\frac{K}{s^2+(K+16)s+60+5K}$$

2. (a) To get the transfer function we ignore the disturbance input (*i.e.* let $N(s) = 0$). Therefore we can write the transfer function as

$$H(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

Therefore

$$\begin{aligned} H(s) &= \frac{\frac{K}{s+10}\frac{1}{s+5}}{1 + \frac{K}{s+10}\frac{1}{s+5}} \\ &= \frac{K}{s^2 + 15s + 50 + K} \end{aligned}$$

(b) The steady state error to an input signal $X(s)$ is given by

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s\left(X(s) - Y(s)\right) \\ &= \lim_{s \to 0} sX(s)\left(1 - H(s)\right) \end{aligned}$$

We have a step input so $X(s) = 1/s$. Therefore

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s\frac{1}{s}\left(1 - H(s)\right) \\ &= \lim_{s \to 0} \left(1 - H(s)\right) \\ &= \lim_{s \to 0} \left(1 - \frac{K}{s^2 + 15s + 50 + K}\right) \\ &= 1 - \frac{K}{50 + K} = \frac{50}{50 + K} \end{aligned}$$

(c) For a ramp input we have $X(s) = 1/s^2$ then

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s\frac{1}{s^2}\left(1 - H(s)\right) \\ &= \lim_{s \to 0} \frac{1}{s}\left(1 - \frac{K}{s^2 + 15s + 50 + K}\right) \\ &= \lim_{s \to 0} \frac{1}{0}\left(1 - \frac{K}{50 + K}\right) \\ &= \infty \end{aligned}$$

(d) To get the disturbance transfer function we let $X(s) = 0$ and find $Y(s)/N(s)$.

$$\begin{aligned} Y(s) &= \frac{1}{s+5}\left(N(s) + \frac{K}{s+10}(X(s) - Y(s))\right) \\ &= \frac{1}{s+5}\left(N(s) - \frac{K}{s+10}Y(s)\right) \\ &= \frac{1}{s+5}N(s) - \frac{K}{(s+10)(s+5)}Y(s) \end{aligned}$$

Re-arranging the equation we get

$$F(s) = \frac{Y(s)}{N(s)} = \frac{\frac{1}{s+5}}{1 + \frac{K}{(s+10)(s+5)}}$$
$$= \frac{s+10}{s^2 + 15s + 50 + K}$$

(e) The steady state disturbance error is given by the net change in the output $Y(s)$ due to the disturbance $N(s)$. Since the effect of the disturbance on the output is $Y(s) = N(s)F(s)$ we can write the disturbance error as

$$n_{ss} = \lim_{s \to 0} sN(s)F(s)$$

For a unit step disturbance

$$n_{ss} = \lim_{s \to 0} s\frac{1}{s}\frac{s+10}{s^2 + 15s + 50 + K}$$
$$= \lim_{s \to 0} \frac{s+10}{s^2 + 15s + 50 + K}$$
$$= \frac{10}{50 + K}$$

3. To find the value of K we first need to find the transfer function and use it to evaluate the steady-state error criterion.

$$
\begin{aligned}
H(s) &= \frac{\frac{100K}{(s+2)(s+5)}}{1+\frac{100K}{(s+2)(s+5)}} \\
&= \frac{100K}{s^2+7s+10+100K}
\end{aligned}
$$

For a steady state error to a unit step input of 1%, $e_{ss} = 0.01$.

$$
\begin{aligned}
\Rightarrow e_{ss} = \lim_{s\to 0} sX(s)\left(1-H(s)\right) &= 0.01 \\
\Rightarrow \lim_{s\to 0} s\frac{1}{s}\left(1-\frac{100K}{s^2+7s+10+100K}\right) &= 0.01 \\
\Rightarrow \lim_{s\to 0}\left(1-\frac{100K}{s^2+7s+10+100K}\right) &= 0.01 \\
\Rightarrow 1-\frac{100K}{10+100K} &= 0.01 \\
\Rightarrow K &= 9.9
\end{aligned}
$$

(a) The form of $H(s)$ is a 2^{nd} order system without finite zeros. Examining the denominator of $H(s)$ and comparing it to the standard form of the 2^{nd} order system transfer function,

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},$$

we see that

$$2\zeta\omega_n = 7$$

and

$$\omega_n^2 = 10+100K.$$

To find the 2% settling time we note that

$$t_s = -\frac{\ln(0.02)}{\zeta\omega_n} = \frac{4}{\zeta\omega_n}.$$

Therefore,

$$t_s = \frac{4}{3.5} = 1.143\text{seconds}.$$

(b) The percentage overshoot is

$$\text{P.O.} = 100\exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

We have

$$\omega_n^2 = 10+100K.$$

Therefore for a steady state error of 1%,

$$\omega_n^2 = 10 + 100(9.9) = 1000.$$

Then $\zeta = 7/(2\sqrt{1000}) \approx 0.11$ and the overshoot

$$\text{P.O.} = 100 \exp\left(-\frac{0.11\pi}{\sqrt{1 - 0.11^2}}\right) \approx 70.5\%$$

4. For this question it is important to remember the relationship between the parameters of the 2^{nd} order system parameters and the pole locations. If we have a 2nd order System with two complex poles and a transfer function

$$H(s) = \frac{n(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

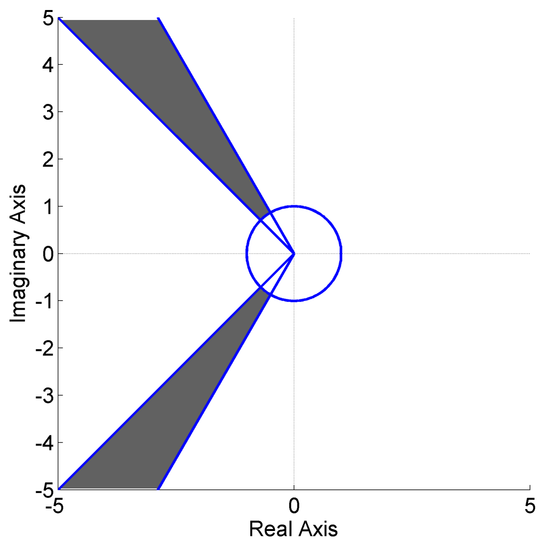
then the poles have a value

$$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

Furthermore the magnitude of the complex pole is ω_n and the angle θ between the complex pole and the negative real axis is $\pm\cos^{-1}(\zeta)$.

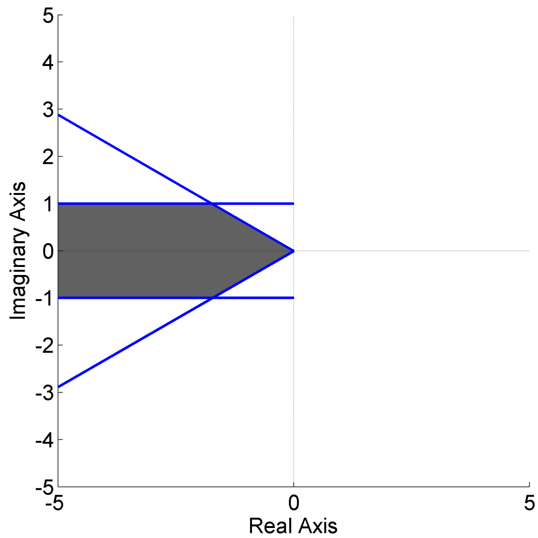
(a) $0.5 < \zeta < 1/\sqrt{2}$ **and** $\omega_n > 1$**.**

The constraints on the pole locations are that $\pi/3 < \theta < \pi/4$, $-\pi/3 > \theta > -\pi/4$ and the magnitude is > 1. We thus shade in the equivalent region on the complex plane



(b) $\zeta < \sqrt{3}/2$ **and** $\omega_n\sqrt{1-\zeta^2} < 1$

The constraints are that $\theta < \pi/6$ $\theta < \pi/6$ and that the imaginary part of the pole is < 1 and > -1



(c) **A 2% settling time < 2 and a percentage overshoot of $< 50\%$.**

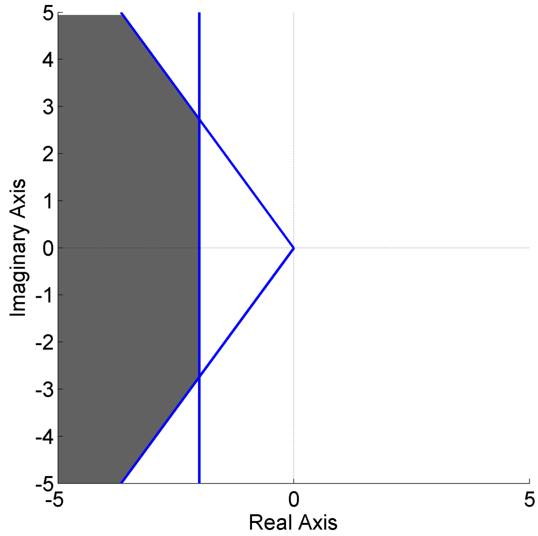
A 2% settling time of < 2 implies that

$$\frac{4}{\zeta\omega_n} < 2$$
$$\Rightarrow \zeta\omega_n > 2$$

Therefore the real part of the pole $= -\zeta\omega_n < 2$.

We have an overshoot of $< 50\%$.

$$\Rightarrow \exp\left\{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right\} < 0.5$$
$$\Rightarrow \zeta > 0.591$$
$$\Rightarrow -53.76^\circ < \theta < 53.76^\circ$$

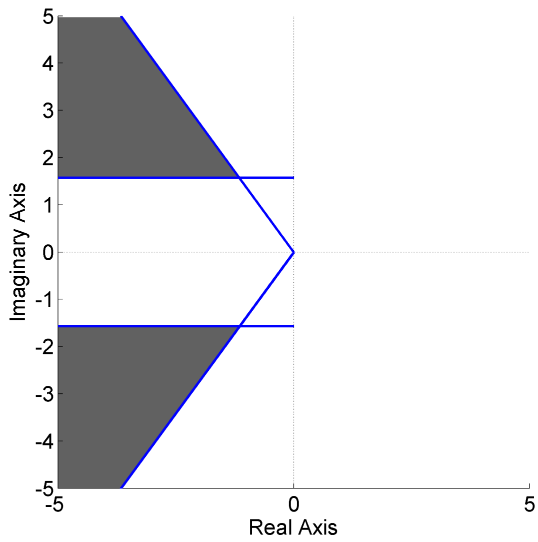


(d) **An overshoot of less than 50% and a peak time of less than 2 seconds.**

The overshoot criteria is the same as part (c) so we have $-53.76^\circ < \theta < 53.76^\circ$. For a peak time of less than 2 seconds we have

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} < 2$$
$$\Rightarrow \omega_n\sqrt{1-\zeta^2} > \pi/2$$

Therefore the imaginary part of the pole is $> \pi/2$ or $< -\pi/2$.



5. The strategy is to complete the steps in the root locus procedure from the notes.

(a) $KP(s) = C(s)G(s) = \frac{K}{s(s+5)(s+2)}$ for $0 < K < \infty$

Identify the start and end points The start points are the poles of $P(s)$ which are at $s = 0$, $s = -5$ and $s = -2$. We have no finite zeros so the 3 plots tend to zeros at infinity.

Identify the potions of the real axis on the root locus The portions are to the left of an odd number of poles and zeros. We have 3 poles on the real axis at 0, -2 and -5 so the interval between 0 and -2 and -5 and $-infty$ are on the root locus.

Identify the centre and angles for the asymptotes to the zeros at infinity . The centre σ_A is given by

$$\begin{aligned} \sigma_A &= \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\text{number of poles} - \text{number of zeros}} \\ &= \frac{0-2-5}{3} = \frac{-7}{3} \end{aligned}$$

The angle of the asymptotes are at

$$\begin{aligned} \phi &= \frac{2k+1}{N-M}\pi \text{ for } k = 0, 1, \ldots, N-M-1 \\ &= \pi/3,\ \pi,\ 5\pi/3 \end{aligned}$$

Find the breakaway points if they exist In this example there must be a breakaway point between 0 and -2 as that portion of the locus is bookended by 2 poles. We have the characteristic equation

$$\begin{aligned} 1 + \frac{K}{s(s+5)(s+2)} &= 0 \\ \Rightarrow K = -s(s+5)(s+2) \end{aligned} \tag{2}$$

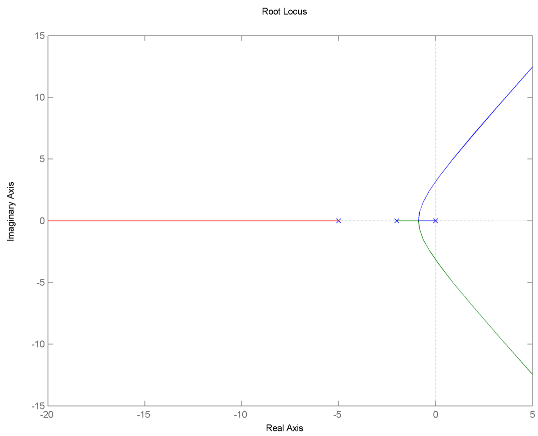
The breakaway point occurs when $\frac{dK}{ds} = 0$ therefore

$$\begin{aligned} \frac{d}{ds}\left(-s(s+2)(s+5)\right) &= 0 \\ \Rightarrow \frac{d}{ds}(s^3 + 7s^2 + 5s) &= 0 \\ \Rightarrow 3s^2 + 14s + 5 &= 0 \\ \Rightarrow s \approx -0.38,\ s &\approx -4.27 \end{aligned}$$

As only root at $s \approx -0.38$, lies in the interval between 0 and -2 it is the location of the breakaway point.

Identify the angle of departure/arrival at complex poles/zeros Here we have no complex poles or zeros so this step can be skipped.

Sketch the Root Locus The root locus is shown below.



(b) $KP(s) = C(s)G(s) = \frac{K}{(s^2+2s+2)(s+1)}$ for $0 < K < \infty$

Identify the start and end points The start points are the poles of $P(s)$ which are at $s = -1$, $s = -1 + j$ and $s = -1 - j$. We have no finite zeros so the 3 plots tend to zeros at infinity.

Identify the potions of the real axis on the root locus We only have one pole on the imaginary axis at $s = -1$ so the root locus is on the real axis in the interval between -1 and $-\infty$.

Identify the centre and angles for the asymptotes to the zeros at infinity The centre σ_A is given by

$$\begin{aligned}\sigma_A &= \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\text{number of poles} - \text{number of zeros}} \\ &= \frac{-1-1+j-1-j}{3} = -1\end{aligned}$$

The angle of the asymptotes are at

$$\begin{aligned}\phi &= \frac{2k+1}{N-M}\pi \text{ for } k = 0, 1, \ldots, N-M-1 \\ &= \pi/3,\ \pi,\ 5\pi/3\end{aligned}$$

Find the breakaway points if they exist There are no breakaway points in this example as the only portion on the locus of the real axis is bookended by a pole and a zero.

Identify the angle of departure/arrival at complex poles/zeros In this example there are two complex poles. The angle departure θ at a pole p is given by

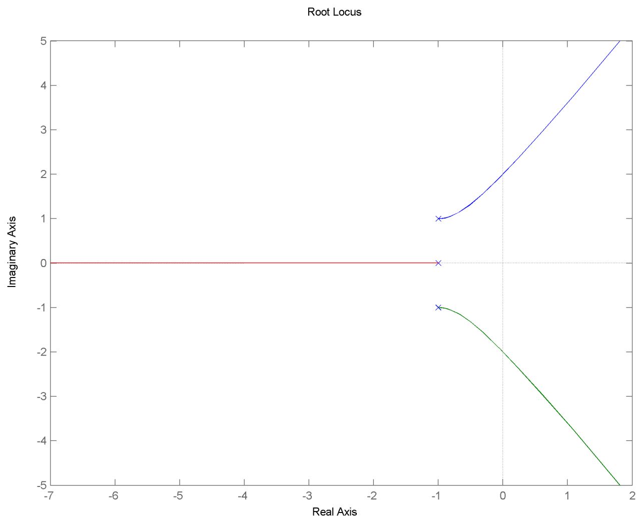
$$\angle(s-p)P(s)\Big|_{s=p} - \theta = \pi$$

For $s = -1 + j$ we have

$$\begin{aligned}\angle(s+1-j)\frac{1}{(s+1)(s+1-j)(s+1+j)}\Big|_{s=-1+j} - \theta &= \pi \\ \Rightarrow \angle\frac{1}{(j)(2j)} - \theta &= \pi \\ \Rightarrow \angle-\frac{1}{2} - \theta &= \pi \\ \Rightarrow \pi - \theta &= \pi \\ \Rightarrow \theta &= 0\end{aligned}$$

For $s = -1 - j$ the angle is $-0 = 0$.

Sketch the Root Locus The root locus is shown below.



(c) $C(s)G(s) = \frac{3(s+K)}{s^2+Ks+2}$ for $0 < K < \infty$

First of all we have to estimate $P(s)$ in the characteristic equation. As the denominator of $H(s) = 1 + C(s)G(s)$ we have a characteristic equation

$$\begin{aligned} 1 + \frac{3(s+K)}{s^2+Ks+2} &= 0 \\ s^2 + Ks + 2 + 3s + 3K &= 0 \\ s^2 + 3s + 2 + K(s+3) &= 0 \\ 1 + \frac{K(s+3)}{s^2+3s+2} &= 0 \end{aligned}$$

Identify the start and end points We have 2 poles at $s = -1$ and $s = -2$ and one zero at $s = -3$. The second root locus branch will terminate at a zero at infinity.

Identify the potions of the real axis on the root locus We have two poles and a zeros on the real axis. Therefore the intervals on the real axis on the root locus are between -1 and -2 and between -3 and $-\infty$.

Identify the centre and angles for the asymptotes to the zeros at infinity We have only one zero at infinity and we know from the previous stage that the asymptote is the negative real axis (*i.e.* $\phi = \pi$).

Find the breakaway points if they exist In this example we have two breakaway points as the interval between -1 and -2 is bookended by two poles and the interval between -3 and $-\infty$ is bookended by two zeros. We have the characteristic equation

$$\begin{aligned} 1 + \frac{K(s+3)}{s^2+3s+2} &= 0 \\ \Rightarrow K &= -\frac{s^2+3s+2}{s+3} \end{aligned} \qquad (3)$$

The breakaway point occurs when $\frac{dK}{ds} = 0$ therefore

$$\begin{aligned} \frac{d}{ds}\left(-\frac{s^2+3s+2}{s+3}\right) &= 0 \\ \Rightarrow -\frac{(s+3)(2s+3) - (s^2+3s+2)}{(s+3)^2} &= 0 \\ \Rightarrow -\frac{s^2+6s+7}{(s+3)^2} &= 0 \\ \Rightarrow s \approx -4.41,\ s &\approx -1.59 \end{aligned}$$

As we have two breakaway points and the roots lie in the appropriate intervals the roots correspond to the breakaway point locations.

Identify the angle of departure/arrival at complex poles/zeros Here we have no complex poles or zeros so this step can be skipped.

Sketch the Root Locus The root locus is shown below.